The existence of NBIBDs with \( k_1 = 6 \) and \( \lambda_1 = 5 \)

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Abstract

In this paper we focus on the existence of nested balanced incomplete block designs with \( k_1 = 6 \) and \( \lambda_1 = 5 \). The necessary conditions for the existences of a \((v, 6, 2, 5, 1)\)-NBIBD and a \((v, 6, 3, 5, 2)\)-NBIBD are the same, i.e., \( v \equiv 0, 1 \pmod{3} \). For \((v, 6, 2, 5, 1)\)-NBIBD, the necessary conditions are also sufficient. For \((v, 6, 3, 5, 2)\)-NBIBD, the necessary conditions are sufficient except for \( v \in \{6, 10\} \), and possibly for \( v \in \{22, 34, 39, 45, 48, 51, 54, 60, 64, 69, 70, 72, 75, 82, 87, 88, 90, 102, 111, 148\} \).

Keywords: nested balanced incomplete block design; perfect Mendelsohn design; generalized whist tournament design
Mathematics Subject Classification: 05B07

1 Introduction

A \((v, k, \lambda)\)-balanced incomplete block design (briefly \((v, k, \lambda)\)-BIBD) is a pair \((V, \mathcal{B})\), where \( V \) is a set of \( v \) elements, \( \mathcal{B} \) is a collection of \( k \)-subsets, called blocks, of \( V \) such that every pair of distinct elements of \( V \) occurs in exactly \( \lambda \) blocks of \( \mathcal{B} \).

Let \( k_1 \) be a positive integer. Let \( k_2 \) be a submultiple of \( k_1 \) and \( k_2 > 1 \). If each block of a \((v, k_1, \lambda_1)\)-BIBD \((V, \mathcal{B}_1)\) can be partitioned into \( k_1/k_2 \) sub-blocks

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of size $k_2$, and the collection of all sub-blocks constitutes a $(v, k_2, \lambda_2)$-BIBD $(V, B_2)$, then the triple $(V, B_1, B_2)$ is said to be a nested balanced incomplete block design, denoted by a $(v, k_1, k_2, \lambda_1, \lambda_2)$-NBIBD.

The blocks of a NBIBD will be displayed in the form of $\{x_1, \ldots, x_{k_2}; y_1, \ldots, y_{k_2}; \ldots\}$ with the semicolons separating the sub-blocks. It is easy to verify that the necessary conditions for the existence of a $(v, k_1, k_2, \lambda_1, \lambda_2)$-NBIBD are $k_2 | k_1$, $\lambda_1(v - 1) \equiv 0 \pmod{(k_1 - 1)}$, $\lambda_2(v - 1) \equiv 0 \pmod{(k_2 - 1)}$, $\lambda_1v(v - 1) \equiv 0 \pmod{k_1(k_1 - 1)}$, $\lambda_2(v - 1) \equiv 0 \pmod{k_2(k_2 - 1)}$, $\lambda_2(1 - k_1) = \lambda_1(1 - k_2)$. NBIBDs were first introduced in 1967 by Preece [10]. Its study has been motivated by a biological experimental background on the effect of inoculating plants with virus [6, 7]. For more information on NBIBDs, the interested reader may refer to [8, 9] and the references therein.

NBIBDs are closely related to generalized whist tournament designs. A generalized whist tournament design is a resolvable BIBD. A BIBD is resolvable if its blocks can be partitioned into parallel classes; a parallel class is a set able $(v, k, \lambda)$-GWhD (briefly $(v, k)$-GWhD) which satisfies (1) $H$ is a partition of $v$-set $X$ into subsets called holes; (2) $B$ is a family of cyclically ordered $k$-subsets of $X$ (called blocks) such that a hole and a block contain at most one common block; (3) every ordered pair of elements from distinct holes are $t$-apart in exactly $\lambda$ blocks for each $t = 1, 2, \ldots, k - 1$. If $H$ contains $u_i$ groups of size $g_i$ for $1 \leq i \leq r$, then we call $g_1^{u_1}g_2^{u_2}\cdots g_r^{u_r}$ the group type (or type) of the HPMD. A $(v, k, \lambda)$-HPMD of type $1^{v-n_1}$ is called an incomplete perfect Mendelsohn design, denoted by a $(v, n, k, \lambda)$-IPMD. A $(v, k, \lambda)$-IPMD of type $1^r$ is simply written as a $(v, k, \lambda)$-PMD.

**Lemma 1.1** ([5]) A $(t, k)$GWhD is a $(v, k, t, k - 1, t - 1)$-NBIBD with the additional property of resolvability or near-resolvability.

NBIBDs are also closely related to perfect Mendelsohn designs. A set of $k$ distinct elements $\{a_1, a_2, \ldots, a_k\}$ is said to be cyclically ordered by $a_1 < a_2 < \cdots < a_k < a_1$ and the elements $a_i, a_{i+t}$ are said to be $t$-apart in a cyclic $k$-tuple $(a_1, a_2, \ldots, a_k)$, where $i + t$ is taken modulo $k$. A $(v, k, \lambda)$-holey perfect Mendelsohn design (briefly $(v, k, \lambda)$-HPMD) is a triple $(X, H, B)$ which satisfies (1) $H$ is a partition of $v$-set $X$ into subsets called holes; (2) $B$ is a family of cyclically ordered $k$-subsets of $X$ (called blocks) such that a hole and a block contain at most one common block; (3) every ordered pair of elements from distinct holes are $t$-apart in exactly $\lambda$ blocks for each $t = 1, 2, \ldots, k - 1$. If $H$ contains $u_i$ groups of size $g_i$ for $1 \leq i \leq r$, then we call $g_1^{u_1}g_2^{u_2}\cdots g_r^{u_r}$ the group type (or type) of the HPMD. A $(v, k, \lambda)$-HPMD of type $1^{v-n_1}$ is called an incomplete perfect Mendelsohn design, denoted by a $(v, n, k, \lambda)$-IPMD. A $(v, k, \lambda)$-IPMD of type $1^r$ is simply written as a $(v, k, \lambda)$-PMD.

**Lemma 1.2** Suppose that there exists a $(v, k, 1)$-PMD, then there exists a $(v, k, t, k - 1, t - 1)$-NBIBD for each $t$ satisfying $t | k$. 

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**Proof** Assume that there exists a $(v, k, 1)$-PMD. Let $k = et$ and $(a_1, \ldots, a_k)$ be a block of the $(v, k, 1)$-PMD. Rearrange this block as follows:

$$\{a_1, a_1 + e, \ldots, a_1 + (t-1)e; a_2, a_2 + e, \ldots, a_2 + (t-1)e; \ldots; a_e, a_2e, \ldots, a_k\}.$$ 

Apply the above procedure to each block of the PMD. Then we have a $(v, k, t, k-1, t-1)$-NBIBD. \(\square\)

We quote the following results for later use.

**Theorem 1.3** ([1, Theorem 10.1]) The necessary conditions for the existence of a $(v, 6, 1)$-PMD, namely, $v \equiv 0, 1 \pmod{3}$ and $v \geq 6$, are sufficient except for $v = 6$, and possibly for $v \in \{12, 18, 24, 30, 36, 40, 45, 50, 54, 60, 69, 72, 75, 80, 84, 88, 90, 96, 100, 102, 104, 108, 114, 116, 120, 122, 126, 130, 132, 134, 138, 140, 142, 144, 146, 150, 152, 154, 156, 158, 160, 162, 164, 166, 168, 170, 172, 174, 176, 178, 180, 182, 184, 186, 188, 190, 192, 194, 196, 198\};

(2) $v \equiv 3 \pmod{6}$ and $v \in \{9, 135\} \cup \{153, 183\} \cup \{207, 213, 219, 237, 243, 255, 297, 375, 411, 435, 453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\};

(3) $v \equiv 4 \pmod{6}$ and $v \in \{10, 16, 22, 34\} \cup \{52, 148\}$.

**Theorem 1.4** ([1, Theorem 3.4]) Suppose that $q = 4t + 1$ is a prime power.

(1) If $t$ is odd and $t \neq 1$, then there exists a $(q + t, t, 6, 1)$-IPMD.

(2) If $t \equiv 2 \pmod{4}$ and $t \neq 2$, then there exists a $(q + t, t, 6, 1)$-IPMD.

(3) If $t \equiv 4 \pmod{8}$, then there exists a $(q + t, t, 6, 1)$-IPMD.

In this paper, we shall focus our attention on the problem of the existence of NBIBDs with $k_1 = 6$, $\lambda_1 = 5$. It is easy to verify that the necessary condition for the existence of a $(v, 6, 2, 5, 1)$-NBIBD and a $(v, 6, 3, 5, 2)$-NBIBD is $v \equiv 0, 1 \pmod{3}$. As the main results, we are to prove the following theorems.

**Theorem 1.5** The necessary condition for the existence of a $(v, 6, 2, 5, 1)$-NBIBD, namely, $v \equiv 0, 1 \pmod{3}$ and $v \geq 6$, is also sufficient.

**Theorem 1.6** The necessary condition for the existence of a $(v, 6, 3, 5, 2)$-NBIBD, namely, $v \equiv 0, 1 \pmod{3}$ and $v \geq 6$, is also sufficient, except for $v \in \{6, 10\}$, and possibly for $v \in \{22, 34, 39, 45, 48, 51, 54, 60, 64, 69, 70, 72, 75, 82, 87, 88, 90, 102, 111, 148\}$.
2 Basic design constructions

In this section we shall give some recursive constructions for NBIBDs. First
we need to introduce some auxiliary designs.

Let $X$ be a set of $v$ elements. A $(v, k_1, k_2, \lambda_1, \lambda_2)$-nested group divisible de-
sign (briefly, $(v, k_1, k_2, \lambda_1, \lambda_2)$-NGDD) is a quadruple $(X, \mathcal{H}, \mathcal{B}_1, \mathcal{B}_2)$ satisfying the
following properties: (1) $\mathcal{H}$ is a partition of $X$ into subsets (called groups
or holes); (2) $\mathcal{B}_1$ is a collection of subsets of $X$ (called blocks), each of size $k_1$;
(3) each of blocks of $\mathcal{B}_1$ can be partitioned into $k_1/k_2$ sub-blocks of size $k_2$, and
denote the collection of all sub-blocks of $\mathcal{B}_1$ by $\mathcal{B}_2$; (4) every 2-subset of $X$ is
either contained in exactly $\lambda_1$ block of $\mathcal{B}_1$ or in exactly one group, but not in
both; (5) every 2-subset of $X$ is either contained in exactly $\lambda_2$ block of $\mathcal{B}_2$ or
in exactly one group, but not in both. If $\mathcal{H}$ contains $u_i$ groups of size $g_i$ for
$1 \leq i \leq r$, then we call $g_1^{u_1}g_2^{u_2} \cdots g_r^{u_r}$ the group type (or type)
of the NGDD. Especially a $(v, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $1^{v-n}$ is called an incomplete nested
balanced incomplete block design, denoted by $(v, n, k_1, k_2, \lambda_1, \lambda_2)$-INBIBD.

The following construction is simple but very useful.

**Construction 2.1** (Filling construction) Suppose that there is a $(v, k_1, k_2, \lambda_1,
\lambda_2)$-NGDD of type $\{m_1, m_2, \ldots, m_s\}$, where $v = \sum_{i=1}^{s} m_i$. If there exist an
$(m_i + w, w, k_1, k_2, \lambda_1, \lambda_2)$-INBIBD for each $1 \leq i \leq s - 1$, and an $(m_s +
w, k_1, k_2, \lambda_1, \lambda_2)$-NBIBD, then there is a $(v + w, k_1, k_2, \lambda_1, \lambda_2)$-NBIBD.

Let $K$ be a set of positive integers. A group divisible design (GDD) $K$-GDD
is a triple $(X, \mathcal{G}, \mathcal{A})$ satisfying the following properties: (1) $\mathcal{G}$ is a partition of
a finite set $X$ into subsets (called groups); (2) $\mathcal{A}$ is a set of subsets of $X$ (called
blocks), each of cardinality from $K$, such that every 2-subset of $X$ is either
contained in exactly one block or in exactly one group, but not in both. If $\mathcal{G}$
contains $u_i$ groups of size $g_i$ for $1 \leq i \leq r$, then we call $g_1^{u_1}g_2^{u_2} \cdots g_r^{u_r}$ the group
type (or type) of the GDD. If $K = \{k\}$, we write $\{k\}$-GDD as $k$-GDD.

A transversal design (TD) $TD(k, n)$ is a GDD of group type $n^k$ and block
size $k$. It is well known that a $TD(k, n)$ is equivalent to $k - 2$ mutually or-
thogonal Latin squares (MOLS) of order $n$. The following lemma will be used
later.

**Lemma 2.2** [2]

(1) A $TD(6, m)$ exists for all integers $m > 4$ except for $m = 6$ and possibly
for $m \in \{10, 14, 18, 22\}$.

(2) A $TD(q + 1, q)$ exists for any prime power $q$.

In recursive constructions of GDDs and PBDs, the weighting technique and
Wilson’s Fundamental Construction [11] are frequently used. Similar tech-
niques are also available for constructing NGDDs.
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**Construction 2.3** (Weighting construction) Suppose that $(X, G, B)$ is a $K$-GDD and $w$ is a function from $X$ to $Z^+ \cup \{0\}$. Suppose that there exists a $\sum_{x \in B} w(x), k_1, k_2, \lambda_1, \lambda_2$-NGDD of type $\{w(x)|x \in B\}$ for every $B \in B$. Then there exists a $(v, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $\{\sum_{x \in G} w(x)|G \in G\}$, where $v = \sum_{x \in X} w(x)$.

**Proof** For every $x \in X$, let $S(x)$ be a set of $\omega(x)$ “copies” of $x$. For every block $B \in B$, construct a $(\sum_{x \in B} w(x), k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $\{w(x)|x \in B\}$ $(\bigcup_{x \in B} S_x, \{S_x|x \in B\}, A_B, C_B)$. Denote

$$ Y = \bigcup_{x \in X} S_x, \quad \mathcal{H} = \bigcup_{x \in G} S_x|G \in G\}, \quad A = \bigcup_{B \in B} A_B, \quad C = \bigcup_{B \in B} C_B. $$

Then it is readily checked that $(Y, \mathcal{H}, A, C)$ is the required NGDD.

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**Construction 2.4** (Inflating construction) If there exist a $(\sum_{i=1}^s g_i h_i, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $g_1h_1g_2h_2 \cdots g_s h_s$ and a TD$(k_1, m)$, then an $m(\sum_{i=1}^s g_i h_i, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $(g_1h_1g_2h_2 \cdots g_s h_s)$ exists.

**Proof** Let $(X, G, B_1, B_2)$ be a $(\sum_{i=1}^s g_i h_i, k_1, k_2, \lambda_1, \lambda_2)$-NGDD of type $g_1h_1g_2h_2 \cdots g_s h_s$. Denote $M = \{1, 2, \ldots, m\}$. Let $X^* = X \times M$. For each $B \in B_1$, construct a TD$(k_1, m)$ on $B \times M$, whose block set is denoted by $A_B$. Let $B = \bigcup_{j=1}^{k_1/k_2} B_j$ and $B_j \in B_2$. Then for each $1 \leq j \leq k_1/k_2$, we have a TD$(k_2, m)$ on $B_j \times M$, which is obtained by removing elements of $(B \setminus B_i) \times M$ in the TD$(k_1, m)$. Denote its block set by $C^{(j)}$. Let $C_B = \bigcup_{j=1}^{k_1/k_2} C^{(j)}$. Let

$$ \mathcal{H} = \bigcup_{x \in G} (\{x\} \times M)|G \in G\}, \quad A = \bigcup_{B \in B_1} A_B, \quad C = \bigcup_{B \in B_1} C_B. $$

It is readily checked that $(X^*, \mathcal{H}, A, C)$ is the required NGDD.

Similar arguments to that in Lemma 1.2, we have

**Lemma 2.5** If there exists a $(v, k, 1)$-HPMD of type $g_1h_1g_2h_2 \cdots g_s h_s$, then there exists a $(v, k, t, k - 1, t - 1)$-NGDD of type $g_1h_1g_2h_2 \cdots g_s h_s$ for each $t$ satisfying $t$ $|$ $k$.

### 3 Existence of $(v, 6, 2, 5, 1)$-NBIBDs

**Lemma 3.1** [9] There exists a $(v, 6, 2, 5, 1)$-NBIBD for $v \in \{6, 7, 9, 10, 12, 13, 15, 16\}$.

By Lemma 1.1, a $(t,k)$GWhD($v$) implies a $(v,k,t,k - 1,t - 1)$-NBIBD. Since for any positive integer $v \equiv 0 \pmod{6}$ and $v \not\equiv \{18, 108, 132, 174, 264\}$, a $(2,6)$GWhD($v$) exists [3], we have

...
Lemma 3.2  There exists a \((v, 6, 2, 5, 1)\)-NBIBD for any positive integer \(v \equiv 0 \pmod{6}\) and \(v \not\in \{18, 108, 132, 174, 264\}\).

By Lemma 1.2, if there exists a \((v, k, 1)\)-PMD, then there exists a \((v, k, t, k−1, t−1)\)-NBIBD for each \(t\) satisfying \(t \mid k\). Combining the results of Theorem 1.3, Lemmas 3.1 and 3.2, we have

Lemma 3.3  There is a \((v, 6, 2, 5, 1)\)-NBIBD for any integer \(v \equiv 0, 1 \pmod{3}\) and \(v \geq 6\) except for

(1) \(v \in \{18, 108, 132\}\);

(2) \(v \equiv 3 \pmod{6}\) and \(v \in [21, 135] \cup [153, 183] \cup \{207, 213, 219, 237, 243, 255, 297, 375, 411, 435, 453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\}\);

(3) \(v \equiv 4 \pmod{6}\) and \(v \in \{22, 34\} \cup [52, 148]\).

Lemma 3.4  There exists a \((v, 6, 2, 5, 1)\)-NBIBD for \(v \in \{18, 21, 22, 27, 33, 34, 39\}\).

Proof  For \(v \in \{21, 22, 33, 39\}\), let \(X = Z_v\) and develop the following base blocks:

\[
\begin{align*}
v = 21: & \quad \{0, 2; 6, 15; 12, 16\}, \quad \{0, 7; 11, 19; 3, 6\}, \quad \{0, 15; 16, 17; 7, 18\}, \\
& \quad \{0, 16; 2, 7; 9, 14\}^*, \quad \{0, 2; 3, 18; 16, 6\}, \quad \{0, 8; 6, 15; 1, 5\}, \quad \{+1 \text{ mod } 21\}; \\
v = 22: & \quad \{0, 5; 1, 2; 10, 13\}, \quad \{0, 6; 11, 17; 2, 13\}^*, \quad \{0, 28; 2, 16; 3, 29\}, \quad \{0, 16; 2, 3; 27, 12\}, \quad \{0, 20; 26, 16; 28, 30\}, \quad \{0, 21; 20, 28; 6, 30\}, \quad \{0, 4; 11, 15; 22, 26\}^*, \quad \{+1 \text{ mod } 33\}; \\
v = 33: & \quad \{0, 3; 12, 18; 17, 28\}, \quad \{0, 28; 2, 16; 3, 29\}, \quad \{0, 16; 2, 3; 27, 12\}, \quad \{0, 20; 26, 16; 28, 30\}, \quad \{0, 21; 20, 28; 6, 30\}, \quad \{0, 4; 11, 15; 22, 26\}^*, \quad \{+1 \text{ mod } 33\}; \\
v = 39: & \quad \{0, 4; 3, 17; 19, 32\}, \quad \{0, 1; 30, 12; 5, 36\}, \quad \{0, 9; 8, 25; 3, 27\}, \quad \{0, 2; 3, 31; 7, 23\}, \quad \{0, 3; 9, 28; 30, 1\}, \quad \{0, 5; 9, 16; 22, 34\}, \quad \{0, 6; 13, 19; 26, 32\}^*, \quad \{+1 \text{ mod } 39\}.
\end{align*}
\]

For \(v \in \{18, 27, 34\}\), let \(X = Z_{v−1} \cup \{\infty\}\) and develop the following base blocks:

\[
\begin{align*}
v = 18: & \quad \{\infty, 12; 0, 9; 2, 8\}, \quad \{0, 15; 1, 2; 4, 7\}, \quad \{0, 12; 1, 8; 3, 7\}, \quad \{+1 \text{ mod } 17\}; \\
v = 27: & \quad \{\infty, 0; 7, 13; 11, 21\}, \quad \{0, 3; 6, 14; 24, 17\}, \quad \{0, 9; 20, 25; 16, 18\}, \quad \{0, 22; 1, 2; 8, 23\}, \quad \{0, 13; 16, 4; 3, 17\}^*, \quad \{+1 \text{ mod } 26\}; \\
v = 34: & \quad \{\infty, 0; 30, 9; 7, 29\}, \quad \{0, 16; 32, 3; 24, 18\}, \quad \{0, 30; 9, 28; 15, 23\}, \quad \{0, 31; 3, 4; 12, 32\}, \quad \{0, 7; 23, 13; 21, 30\}, \quad \{0, 15; 11, 26; 22, 4\}^*, \quad \{+1 \text{ mod } 33\}, \quad \{+1 \text{ mod } 33\};
\end{align*}
\]

where \(\infty + 1 = \infty\). When \(v \in \{21, 33, 39\}\), the base blocks with a * will generate \(v/3\) distinct blocks under \(Z_v\); when \(v = 22\), the base block with a * will generate 11 distinct blocks under \(Z_{22}\); when \(v = 27\), the base block with a * will generate 13 distinct blocks under \(Z_{26}\); when \(v = 34\), the base blocks with a * will generate 11 distinct blocks under \(Z_{33}\). \(\square\)
Lemma 3.5 There exists a $(11, 2, 6, 2, 5, 1)$-INBIBD.

Proof Let $X = Z_9 \cup \{\infty_1, \infty_2\}$ with $\{\infty_1, \infty_2\}$ as the hole. Develop the two base blocks: $\{0, 1, 4, 8, 3, \infty_1\}$ and $\{0, 2, 5, 8, 7, \infty_2\}$ by $+1$ modulo 9 to obtain all the blocks, where $\infty_i + 1 = \infty_i$ for each $i = 1, 2$. \hfill \square

Lemma 3.6 There exists a $(69, 12, 6, 2, 5, 1)$-INBIBD.

Proof Let $X = Z_{57} \cup \{\infty_1, \infty_2, \ldots, \infty_{12}\}$ with $\{\infty_1, \infty_2, \ldots, \infty_{12}\}$ as the hole. Develop the following base blocks by $+1$ modulo 57 to obtain all the blocks, where $\infty_i + 1 = \infty_i$ for each $1 \leq i \leq 12$.

\[
\begin{align*}
\{19, 1; 8, 38; 37, 11\}, & \quad \{0, 55; 33, 24; 40, \infty_1\}, & \quad \{6, 28; 10, 51; 0, \infty_2\}, \\
\{0, 42; 30, 13; 56, \infty_3\}, & \quad \{15, 19; 51, 18; 38, \infty_4\}, & \quad \{0, 32; 42, 30; 37, \infty_5\}, \\
\{27, 33; 19, 38; 6, \infty_6\}, & \quad \{0, 21; 9, 32; 49, \infty_7\}, & \quad \{0, 10; 9, 14; 3, \infty_8\}, \\
\{23, 24; 0, 7, 39, \infty_9\}, & \quad \{0, 11; 21, 13; 9, \infty_{10}\}, & \quad \{0, 29; 9, 52; 3, \infty_{11}\}, \\
\{39, 26; 24, 4; 0, \infty_{12}\}, & \quad \{0, 3; 19, 22; 38, 41\}.
\end{align*}
\]

The base block with a * will generate 19 distinct blocks under $Z_{57}$. \hfill \square

Lemma 3.7 There exists a $(v, 6, 2, 5, 1)$-NBIBD for $v \in \{45, 52, 58, 69\}$.

Proof A $(45, 6, 6, 1)$-IPMD, a $(52, 9, 6, 1)$-IPMD and a 6-HPMD of type $7^{79}4$ are constructed explicitly in [1]. Then by Lemma 2.5, there is a $(45, 6, 6, 2, 5, 1)$-INBIBD, a $(52, 9, 6, 2, 5, 1)$-INBIBD and a $(58, 6, 2, 5, 1)$-NGDD of type $7^{79}4$. For $v = 45$ and 52, fill in the holes with a $(6, 6, 2, 5, 1)$-NBIBD and a $(9, 6, 2, 5, 1)$-NBIBD, which exist by Lemma 3.1. For $v = 58$, fill in the groups with a $(7, 6, 2, 5, 1)$-NBIBD and a $(9, 6, 2, 5, 1)$-NBIBD, which exist by Lemma 3.1. For $v = 69$, by Lemma 3.6, there exists a $(69, 12, 6, 2, 5, 1)$-INBIBD. Then fill in the hole with a $(12, 6, 2, 5, 1)$-NBIBD, which exists by Lemma 3.1. \hfill \square

Lemma 3.8 There exists a $(v, 6, 2, 5, 1)$-NBIBD for $v \in \{51, 57, 63, 64, 70, 81, 82, 105, 106, 108, 111, 112, 117, 129, 130, 135, 136, 148, 171, 177, 243, 375, 411\}$.

Proof Start from a $(u, 6, 2, 5, 1)$-NBIBD, which is also a $(u, 6, 2, 5, 1)$-NGDD of type $1^u$. Apply Construction 2.4 with a TD$(6, m)$ to obtain a $(um, 6, 2, 5, 1)$-NGDD of type $m^u$. Fill in the groups by Construction 2.1 with an $(m + h, h, 6, 2, 5, 1)$-INBIBD and an $(m + h, 6, 2, 5, 1)$-INBIBD. Then we have an $(mu + h, 6, 2, 5, 1)$-NBIBD. Note that when $h \in \{0, 1\}$, an $(m + h, h, 6, 2, 5, 1)$-INBIBD is just an $(m + h, 6, 2, 5, 1)$-NBIBD. Let $v = mu + h$. Take the parameters $(v, u, m, h)$ as follows, and make use of the above method to obtain all the desired NBIBDs, where the needed TDs are from Lemma 2.2, and the needed NBIBDs are from Lemmas 3.3 and 3.4.
Lemma 3.9 There exists a \((v, 6, 2, 5, 1)\)-NBIBD for \(v \in \{75, 76, 87, 88, 93, 94, 99, 100, 118, 123, 124, 132, 142, 153, 159, 165, 183, 207, 213, 219, 237, 255, 297, 435, 453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\}\).

Proof Start from a TD\((u + 1, m)\). Let \(0 \leq a \leq m\). Give each point weight 1 in the first \(u\) groups. In the last group give \(a\) points weight \(x\) and the remaining \(m - a\) points weight 0. If there exist a \((u, 6, 2, 5, 1)\)-NBIBD and a \((u + x, 6, 2, 5, 1)\)-NBIBD, then applying Construction 2.3, we have an \((mu + ax, 6, 2, 5, 1)\)-NBIBD. Fill in the groups by Construction 2.1 with an \((m + h, 6, 2, 5, 1)\)-NBIBD and a \((ax + h, 6, 2, 5, 1)\)-NBIBD. Then we have an \((mu + ax + h, 6, 2, 5, 1)\)-NBIBD. Let \(v = mu + ax + h\). Take the parameters \((v, u, m, a, h, x)\) as follows, and make use of the above method to obtain all the desired NBIBDs, where the needed TDs are from Lemma 2.2, the needed NBIBDs are from Lemmas 3.3 and 3.4, and the needed \((11, 2, 6, 2, 5, 1)\)-INBIBD is from Lemma 3.5.

Combining the results of Lemmas 3.3, 3.4, 3.7, 3.8 and 3.9, we complete the proof of Theorem 1.5.
4 Existence of \((v, 6, 3, 5, 2)\)-NBIBDs

Lemma 4.1 [9] There is no \((v, 6, 3, 5, 2)\)-NBIBD for \(v \in \{6, 10\}\). If \(v \in \{7, 9, 12, 13, 15, 16\}\), a \((v, 6, 3, 5, 2)\)-NBIBD exists.

By Lemma 1.1, a \((t, k)\text{GWhD}(v)\) implies a \((v, k, t, k - 1, t - 1)\)-NBIBD. Since when \(v \in \{24, 30, 84, 96, 162, 168, 180, 192, 198\}\), a \((3, 6)\text{GWhD}(v)\) exists [4], we have

Lemma 4.2 There is a \((v, 6, 3, 5, 2)\)-NBIBD for \(v \in \{24, 30, 84, 96, 162, 168, 180, 192, 198\}\).

By Lemma 1.2, if there exists a \((v, k, 1)\)-PMD, then there exists a \((v, k, t, k - 1, t - 1)\)-NBIBD for each \(t\) satisfying \(t \mid k\). Combining the results of Theorem 1.3, Lemmas 4.1 and 4.2, we have

Lemma 4.3 There is a \((v, 6, 3, 5, 2)\)-NBIBD for any integer \(v \equiv 0, 1\) (mod 3) and \(v \geq 7\) except for

1. \(v \in \{18, 48, 54, 60, 72, 90, 102, 108, 114, 132, 138, 150\}\);
2. \(v \equiv 3\) (mod 6) and \(v \in \{21, 135\} \cup \{153, 183\} \cup \{207, 213, 219, 237, 243, 255, 297, 375, 411, 435, 453, 459, 471, 489, 495, 513, 519, 609, 615, 621, 657\}\);
3. \(v \equiv 4\) (mod 6) and \(v \in \{10, 22, 34\} \cup \{52, 148\}\).

Lemma 4.4 There is a \((v, 6, 3, 5, 2)\)-NBIBD for \(v \in \{18, 21, 27, 33\}\).

Proof For \(v \in \{21, 33\}\), let \(X = Z_v\) and develop the following base blocks:

\[
\begin{align*}
v = 21: & \quad \{0, 11, 19, 6, 8, 7\}, \quad \{0, 3, 8; 15, 4, 19\}, \quad \{0, 3, 9; 10, 1, 6\}, \\
& \quad \{0, 7, 14; 2, 9, 16\}^*, \quad (+1 \text{ mod } 21);
\end{align*}
\]

\[
\begin{align*}
v = 33: & \quad \{0, 6, 12; 13, 22, 30\}, \quad \{0, 10, 23; 29, 25, 32\}, \quad \{0, 1, 5; 17, 30, 32\}, \\
& \quad \{0, 1, 15; 20, 23, 28\}, \quad \{0, 2, 14; 21, 5, 12\}, \quad \{0, 11, 22; 4, 15, 26\}^*, \\
& \quad (+1 \text{ mod } 33).
\end{align*}
\]

For \(v \in \{18, 27\}\), let \(X = Z_{v-1} \cup \{\infty\}\) and develop the following base blocks:

\[
\begin{align*}
v = 18: & \quad \{\infty, 0, 9; 2, 8, 12\}, \quad \{0, 1, 15; 2, 4, 7\}, \quad \{0, 1, 7; 3, 8, 12\}, \\
& \quad (+1 \text{ mod } 17);
\end{align*}
\]

\[
\begin{align*}
v = 27: & \quad \{\infty, 0, 13; 7, 11, 21\}, \quad \{0, 1, 8; 2, 22, 23\}, \quad \{0, 9, 20; 16, 18, 25\}, \\
& \quad \{0, 3, 24; 6, 14, 17\}, \quad \{0, 4, 16; 13, 17, 3\}^*, \quad (+1 \text{ mod } 26),
\end{align*}
\]

where \(\infty + 1 = \infty\). When \(v \in \{21, 33\}\), the base blocks with a * will generate \(v/3\) distinct blocks under \(Z_v\); when \(v = 27\), the base block with a * will generate 13 distinct blocks under \(Z_{26}\). \(\square\)
Lemma 4.5 There is a \((q + t, t, 6, 3, 5, 2)\)-INBIBD for \((q, t) \in \{(9, 2), (13, 3), (17, 4), (61, 15)\}\).

Proof By Theorem 1.4, there exists a \((q + t, t, 6, 1)\)-IPMD for \((q, t) \in \{(13, 3), (17, 4), (61, 15)\}\), which implies a \((q + t, t, 6, 3, 5, 2)\)-INBIBD by Lemma 2.5.

When \((q, t) = (9, 2)\), we construct a \((11, 2, 6, 3, 5, 2)\)-INBIBD on \(Z_9 \cup \{\infty_1, \infty_2\}\) as follows, where \(\{\infty_1, \infty_2\}\) is the hole. Develop the two base blocks: \(\{0, 1, 3; 4, 8, \infty_1\}\) and \(\{0, 2, 5; 7, 8, \infty_2\}\) by +1 modulo 9 to obtain all the blocks, where \(\infty_i + 1 = \infty_i\) for each \(i = 1, 2\). □

Lemma 4.6 There is a \((v, 6, 3, 5, 2)\)-NBIBD for \(v \in \{52, 58, 76, 150, 159, 165\}\).

Proof A \((24, 3, 6, 1)\)-IPMD, a \((52, 9, 6, 1)\)-IPMD, a 6-HPMD of type \(3^{10}\), a 6-HPMD of type \(7^3 9^4\) and a 6-HPMD of type \(21^7 9^4\) are constructed explicitly in [1]. Then by Lemma 2.5, there is a \((24, 3, 6, 3, 5, 2)\)-INBIBD, a \((52, 9, 6, 3, 5, 2)\)-INBIBD, a \((30, 6, 3, 5, 2)\)-NGDD of type \(3^{10}\), a \((58, 6, 3, 5, 2)\)-NGDD of type \(7^3 9^4\) and a \((156, 6, 3, 5, 2)\)-NGDD of type \(21^7 9^4\).

For \(v = 52\), fill in the hole of a \((52, 9, 6, 3, 5, 2)\)-INBIBD, where the needed \((9, 6, 3, 5, 2)\)-NBIBD is from Lemma 4.1. For \(v = 58\), fill in the groups of a \((58, 6, 3, 5, 2)\)-NGDD of type \(7^3 9^4\), where the needed \((7, 6, 3, 5, 2)\)-NBIBD and \((9, 6, 3, 5, 2)\)-NBIBD are from Lemma 4.1. For \(v = 76\), start from a \((76, 15, 6, 3, 5, 2)\)-INBIBD, which exists by Lemma 4.5. Fill in the hole with a \((15, 6, 3, 5, 2)\)-NBIBD, which exists by Lemma 4.1. For \(v = 150\), inflate a \((30, 6, 3, 5, 2)\)-NGDD of type \(3^{10}\) with a TD(6, 5) to obtain a \((150, 6, 3, 5, 2)\)-NGDD of type \(15^{10}\). Fill in the groups with a \((15, 6, 3, 5, 2)\)-NBIBD, which exists by Lemma 4.1. For \(v = 159\), add three infinite points and fill in the groups of a \((156, 6, 3, 5, 2)\)-NGDD of type \(21^7 9^4\), where the needed \((24, 3, 6, 3, 5, 2)\)-INBIBD exists by the first paragraph of this proof, and the needed \((12, 6, 3, 5, 2)\)-NBIBD is from Lemma 4.1. For \(v = 165\), start from a \((11, 6, 3, 5, 2)\)-NGDD of type \(1^9 2^1\), which is also a \((11, 2, 6, 3, 5, 2)\)-INBIBD and exists by Lemma 4.5. Inflate this \((11, 6, 3, 5, 2)\)-NGDD of type \(1^9 2^1\) with a TD(6, 15) to obtain a \((165, 6, 3, 5, 2)\)-NGDD of type \(15^9 30^1\). Fill in the groups with a \((15, 6, 3, 5, 2)\)-NBIBD and a \((30, 6, 3, 5, 2)\)-NBIBD, which exist by Lemmas 4.1 and 4.2. □

Lemma 4.7 There is a \((v, 6, 3, 5, 2)\)-NBIBD for \(v \in \{57, 63, 81, 94, 100, 105, 106, 108, 112, 117, 123, 129, 135, 136, 153, 171, 177, 243, 375, 513, 657\}\).

Proof Start from a \((u, 6, 3, 5, 2)\)-NBIBD, which is also a \((u, 6, 3, 5, 2)\)-NGDD of type \(1^u\). Apply Construction 2.4 with a TD(6, m) to obtain a \((um, 6, 3, 5, 2)\)-NGDD of type \(m^u\). Fill in the groups by Construction 2.1 with an \((m + h, h, 6, 3, 5, 2)\)-INBIBD and an \((m + h, 6, 3, 5, 2)\)-NBIBD. Then we have an
(\(mu + h, 6, 3, 5, 2\))-NBIBD. Note that when \(h \in \{0, 1\}\), an \((m+h, h, 6, 3, 5, 2)\)-INBIBD is just an \((m+h, 6, 3, 5, 2)\)-NBIBD. Let \(v = mu + h\). Take the parameters \((v, u, m, h)\) as follows, and make use of the above method to obtain all the desired NBIBDs, where the needed TDs are from Lemma 2.2, the needed NBIBDs are from Lemmas 4.3 and 4.4, the needed \((16, 3, 6, 3, 5, 2)\)-INBIBD and \((21, 4, 6, 3, 5, 2)\)-INBIBD are from Lemma 4.5.

\[
\begin{align*}
(57, 7, 8, 1), & \quad (63, 7, 9, 0), \quad (81, 9, 9, 0), \quad (94, 7, 13, 3), \\
(100, 9, 11, 1), & \quad (105, 15, 7, 0), \quad (106, 7, 15, 1), \quad (108, 12, 9, 0), \\
(112, 16, 7, 0), & \quad (117, 9, 13, 0), \quad (123, 7, 17, 4), \quad (129, 16, 8, 1), \\
(135, 15, 9, 0), & \quad (136, 9, 15, 1), \quad (153, 19, 8, 1), \quad (171, 9, 19, 0), \\
(177, 16, 11, 1), & \quad (243, 9, 27, 0), \quad (375, 15, 25, 0), \quad (513, 19, 27, 0), \\
(57, 9, 73, 0). &
\end{align*}
\]

□

Lemma 4.8 There is a \((v, 6, 3, 5, 2)\)-NBIBD for \(v \in \{93, 99, 114, 118, 124, 130, 132, 138, 142, 183, 207, 213, 219, 237, 255, 297, 411, 435, 453, 459, 471, 489, 495, 519, 609, 615, 621\}\).

Proof Start from a TD\((u + 1, m)\). Let \(0 \leq a \leq m\). Give each point weight 1 in the first \(u\) groups. In the last group give \(a\) points weight \(x\) and the remaining \(m - a\) points weight 0. If there exist a \((u, 6, 3, 5, 2)\)-NGDD of type \(1^{u}\) (i.e., a \((u, 6, 3, 5, 2)\)-NBIBD) and a \((u + x, 6, 3, 5, 2)\)-NGDD of type \(1^{u}x^{1}\) (i.e., a \((u+x, x, 6, 3, 5, 2)\)-INBIBD), then applying Construction 2.3, we have an \((mu + ax, 6, 3, 5, 2)\)-NGDD of type \(m^{a}x^{1}\). Fill in the groups by Construction 2.1 with an \((m+ax, 6, 3, 5, 2)\)-INBIBD and a \((ax + h, 6, 3, 5, 2)\)-NBIBD. Then we have an \((mu + ax + h, 6, 3, 5, 2)\)-NBIBD. Let \(v = mu + ax + h\). Take the parameters \((v, u, m, a, h, x)\) as follows, and make use of the above method to obtain all the desired NBIBDs in this lemma, where the needed TDs are from Lemma 2.2, the needed NBIBDs are from Lemmas 4.3 and 4.4, and the needed \((11, 2, 6, 3, 5, 2)\)-INBIBD and \((16, 3, 6, 3, 5, 2)\)-INBIBD are from Lemma 4.5.

\[
\begin{align*}
(93, 9, 6, 0, 2), & \quad (99, 9, 9, 0, 2), \quad (114, 9, 11, 7, 1, 2), \\
(118, 9, 11, 9, 1, 2), & \quad (124, 9, 13, 2, 3, 2), \quad (130, 9, 13, 5, 3, 2), \\
(132, 9, 13, 6, 3, 2), & \quad (138, 9, 13, 9, 3, 2), \quad (142, 9, 13, 11, 3, 2), \\
(183, 9, 19, 6, 0, 2), & \quad (207, 9, 19, 18, 0, 2), \quad (213, 12, 17, 8, 1, 1), \\
(219, 12, 17, 14, 1, 1), & \quad (237, 12, 19, 9, 0, 1), \quad (255, 15, 16, 15, 0, 1), \\
(297, 15, 19, 12, 0, 1), & \quad (411, 9, 43, 12, 0, 2), \quad (435, 18, 23, 20, 1, 1), \\
(453, 12, 37, 9, 0, 1), & \quad (459, 12, 37, 15, 0, 1), \quad (471, 12, 37, 27, 0, 1), \\
(489, 13, 36, 7, 0, 3), & \quad (495, 13, 36, 9, 0, 3), \quad (519, 9, 49, 39, 0, 2), \\
(609, 12, 49, 21, 0, 1), & \quad (615, 12, 49, 27, 0, 1), \quad (621, 9, 67, 9, 0, 2). \\
\end{align*}
\]
Combine the results of Lemmas 4.3, 4.4, 4.6, 4.7 and 4.8. Then we have Theorem 1.6.

References


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