T- Reich Mapping in Topological Vector Space-Valued Cone Metric Spaces

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Abstract

The object of this paper is to establish some new fixed point results in topological vector space-valued cone metric spaces, by proving the fixed point theorems for T-Reich and T-Kannan contraction mappings in topological vector space-valued cone metric spaces.

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1 Introduction

Huang and Zhang [5] generalized the notion of metric spaces replacing the set of real numbers by an ordered Banach space. Many authors proved fixed point theorems in cone metric spaces (see, e.g. [5, 6, 12] ) under additional assumption about the underlying cone, such as normality or even regularity. Recently, Rezapour and Hambarani [11] omitted the assumption of normality in cone metric space, which is a milestone in developing fixed point theory in
cone metric space. In papers [1, 8] authors tried to generalize this approach by using cones in topological vector spaces (tvs) instead of Banach space. However, it should be noted that an old result [9] shows that if the underlying cone of an ordered tvs is solid and normal it must be an ordered normed space. So proper generalizations when passing from norm-valued cone metric space can be obtained only in the case of non normal cones. Recently Kadelburg et. al. [8] developed further theory of tvs-cone metric space and proved some fixed point results and common fixed point results in tvs-cone metric space. In this paper we prove some fixed point theorem for T-Reich type mappings and T-Kannan type contraction [6] in tvs-valued cone metric space.

2 Preliminary Notes

Definition 2.1. Let $E$ be a real Hausdorff topological vector space (tvs for short) with the zero vector $\theta$. A nonempty proper and closed subset $P$ of $E$ is called a (convex) cone if $P + P \subset P$, $\lambda P \subset P$ for $\lambda \geq 0$ and $P \cap \{-P\} = \{\theta\}$. We will always assume that $P^0$ is non empty (here $P^0$ denotes the interior of $P$), and such cones are called solid.

Each cone $P$ includes a partial order “$\leq$” on $E$ defined by $x \leq y \iff y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in P^0$. The pair $(E, P)$ is an ordered topological vector space.

Let $P$ be a cone in a real Banach space $E$ then $P$ is called normal, if there exists a constant $K > 0$ such that for all $x, y \in E$ and $0 \leq x \leq y$ implies $\|x\| \leq K \|y\|$. The least positive number $K$ satisfying the above inequality is called the normal constant of $P$.

Proposition 2.2. [8] Let $P$ be a cone in a real tvs $E$. If for $a \in P$ and $a \leq ka$, for some $k \in [0, 1)$ then $a = \theta$.

Definition 2.3. Let $X$ be a nonempty set and $(E, P)$ an ordered tvs. A function $d : X \times X \to E$ is called tvs-cone metric and $(X, d)$ is called tvs-cone metric space, if the following conditions hold:

(a) $\theta \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = \theta \iff x = y$,
(b) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(c) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Let $x \in X$ and $\{x_n\}$ be a sequence in $X$. Then it is said the following:

(a) $\{x_n\}$ tvs-cone converges to $x$, if for every $c \in P^0$, there is a natural number $n_0$ such that $d(x_n, x) \ll c$, for all $n > n_0$. We denote it by $\lim_{n \to \infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.
(b) $\{x_n\}$ is a tvs-cone Cauchy sequence if for every $c \in P^0$ there is a natural
number $n_0$ such that $d(x_n, x_m) \ll c$ for all $n, m > n_0$.
(c) $(X, d)$ is tvs-cone complete if every tvs-cone Cauchy sequence is tvs-cone convergent in $X$.

In further discussion we always assume that $E$ is a real tvs and $P$ is a cone in $E$, and “$\leq$” is partial ordering with respect to $P$.

**Lemma 2.4.** [8] (a) Let $\theta \leq x_n \to \theta$ in $(E, P)$ and $\theta \ll c$. Then there is $n_0$ such that $x_n \ll c$ for every $n > n_0$.
(b) It can happened that $\theta \leq x_n \ll c$ for each $n > n_0$, but $x_n \to \theta$ in $(E, P)$.
(c) It can happened that $x_n \to x, y_n \to y$ in the tvs-cone metric $d$, but that $d(x_n, y_n) \to d(x, y)$ in $(E, P)$.
(d) $\theta \leq u \ll c$ for each $c \in P^0 \Rightarrow u = \theta$.
(e) $x_n \to x, x_n \to y$ (in the tvs-cone metric)$\Rightarrow x = y$.

**Lemma 2.5.** [8] (a) If $u \leq v$ and $v \ll w$, then $u \ll w$,
(b) If $u \ll v$ and $v \leq w$, then $u \ll w$,
(c) If $u \ll v$ and $v \ll w$, then $u \ll w$,
(d) Let $x \in X$, and $\{x_n\}$ and $\{b_n\}$ be two sequences in $X$ and $E$ respectively, $\theta \ll c$ and $\theta \leq d(x_n, x) \leq b_n$ for all $n$. If $b_n \to \theta$, then there is $n_0$ such that $d(x_n, x) \ll c$ for all $n > n_0$.

**Definition 2.6.** Let $T$ and $f$ are two self maps of a tvs-valued cone metric space $X$. Then $(T, f)$ is called a Banach pair, if $fT x = T f x$ for every $x \in F(f)$, where $F(f)$ is the set of all fixed point of $f$.

In further discussion we write “0” in place of zero vector “$\theta$” of $E$.

**Definition 2.7.** Let $(X, d)$ be a tvs-cone metric space and $T, f : X \to X$ satisfy, $d(T f x, T f y) \leq ad(T x, T y) + bd(T x, T f x) + cd(T y, T f y)$ for all $x, y \in X$, where $a, b, c$ are nonnegative constants such that $a+b+c < 1$. Then $f$ is called $T$-Reich mapping.

## 3 Main Results

**Theorem 3.1.** Let $(X, d)$ be a complete tvs-cone metric space and $T, f : X \to X$ and $f$ is $T$-Reich mapping, $T, f$ are continuous, $T$ is injective and sub-sequentially convergent mapping, then $f$ has a fixed point in $X$. Moreover, if $(T, f)$ is a Banach pair then $T$ and $f$ have a unique common fixed point in $X$.

**Proof:** Let $x_0 \in X$ be arbitrary, we define a sequence $\{x_n\}$ by $x_{n+1} = f x_n$, for all $n \geq 0$. Now since $f$ is $T$-Reich mapping hence we have,
\[
d(T f x_n, T f x_{n-1}) \leq ad(T x_n, T x_{n-1}) + bd(T x_n, T f x_n) + cd(T x_n, T f x_{n-1})
\]
\[
d(T x_{n+1}, T x_n) \leq ad(T x_n, T x_{n-1}) + bd(T x_n, T x_{n+1}) + cd(T x_{n-1}, T x_n)
\]
Writing $d_n = d(Tx_{n+1}, Tx_n)$ we have,

$$d_n \leq ad_{n-1} + bd_n + cd_{n-1}$$

$$(1-b)d_n \leq (a + c)d_{n-1}$$

$$d_n \leq \frac{a + c}{1 - b} d_{n-1}$$

$$d_n \leq \lambda d_{n-1}$$

where $\lambda = \frac{a + c}{1 - b} < \frac{a + c}{a + c} < 1$. Hence $\lambda < 1$ and $d_n \leq \lambda^n d_0$, where $d_0 = d(x_1, x_0)$.

Now if $m, n \in \mathbb{N}$ and $m > n$ then we have,

$$d(Tx_n, Tx_m) \leq d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + \cdots + d(Tx_{m-1}, Tx_m)$$

$$d(Tx_n, Tx_m) \leq d_n + d_{n+1} + d_{n+2} + \cdots + d_{m-1}$$

$$d(Tx_n, Tx_m) \leq \lambda^n d_0 + \lambda^{n+1} d_0 + \lambda^{n+2} d_0 + \cdots$$

$$d(Tx_n, Tx_m) \leq \lambda^n d_0 [1 + \lambda + \lambda^2 + \cdots]$$

$$d(Tx_n, Tx_m) \leq \frac{\lambda^n d_0}{1 - \lambda} \to 0 \text{ as } n \to \infty \text{ (since } \lambda < 1).$$

Now using properties (a) of Lemma 2.4, and only the assumption that the underlying cone is solid, we have, for every $e \in P^0$ there is $n_0$ such that $\frac{\lambda^n d_0}{1 - \lambda} < e$ for all $n > n_0$ and by (a) of Lemma 2.5, we conclude that $\{Tx_n\}$ is a Cauchy sequence. Since $X$ is complete we must have $u \in X$, such that $Tx_n \to u$ as $n \to \infty$.

Now since $T$ is sub-sequentially convergent, therefore the sequence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to z \in X$, also $T$ is continuous hence $Tx_{n_k} \to Tz$ as $k \to \infty$, and by the uniqueness of limit in tvs-cone metric space we have $Tz = u$.

Now since $f$ is continuous and $x_{n_k} \to z$ so $fx_{n_k} \to fz$ and by continuity of $T$ we have $Tfx_{n_k} \to Tfz$.

Now we show that $Tfz = fz$. Then we have

$$d(Tfz, Tz) \leq d(Tfz, Tfx_{n_k}) + d(Tfx_{n_k}, Tz)$$

$$\leq ad(Tz, Tx_{n_k}) + bd(Tz, Tfz) + cd(Tx_{n_k}, Tfx_{n_k})$$

$$+d(Tfx_{n_k}, Tz)$$

$$= ad(Tz, Tx_{n_k}) + bd(Tz, Tfz) + cd(Tx_{n_k}, Tx_{n_k+1})$$

$$+d(Tx_{n_k+1}, Tz)$$

$$(1-b)d(Tz, Tfz) \leq ad(Tz, Tx_{n_k}) + cd_{n_k} + d(Tx_{n_k+1}, Tz)$$

$$d(Tz, Tfz) \leq \frac{a}{1-b} d(Tz, Tx_{n_k}) + \frac{c}{1-b} d_{n_k} + \frac{1}{1-b} d(Tx_{n_k+1}, Tz)$$

Now since $Tx_{n_k} \to Tz$ and $d_{n_k} \to 0$ as $k \to \infty$, hence for any given $e \in P^0$ we can choose $n_1$ such that $d(Tz, Tx_{n_k}) \ll \frac{1}{a+e}$, $d_{n_k} \ll \frac{1}{a+e}$ and $d(Tfx_{n_k+1}, Tz) \ll$
1 - b for all $k > n_1$. Hence we have
\[ d(Tz, Tfz) \ll \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = e \]
for all $e \in P^0$.
So we have $d(Tz, Tfz) = 0$. Hence $Tfz = Tz$, but $T$ is injective hence $fz = z$
i.e. $z$ is a fixed point of $f$.

Now we show that $z$ is unique fixed point of $f$. Let $w$ is another fixed point of
then we have $fw = w$ and
\[
d(Tz, Tw) = d(Tfz, Tfw) \\
\leq ad(Tz, Tw) + bd(Tz, Tfz) + cd(Tw, Tfw) \\
= ad(Tz, Tw) + bd(Tz, Tz) + cd(Tw, Tw) \\
= ad(Tz, Tw)
\]

since $0 \leq a < 1$, hence by proposition 2.2, we must have $d(Tz, Tw) = 0$ i.e. $Tz = Tw$, and $T$ is injective hence $z = w$. Thus fixed point is unique.

Now if $(T, f)$ is a Banach pair, then $T$ and $f$ commutes at the fixed point of
which implies that $fTz = Tfz$ i.e. $Tfz = Tz$. It shows that $Tz$ is another
fixed point of $f$. Hence by uniqueness of fixed point of $f$ we must have $Tz = z$
i.e. $z$ is also a fixed point of $T$, and by uniqueness of fixed point of $f$, it is
unique common fixed point of $f$ and $T$. □

The following corollary extends the main result of Beiranvand [2] to the
tvs-cone metric space.

**Corollary 3.2.** (T-contraction) Let $(X, d)$ be a complete tvs-cone metric
space and $T, f : X \to X$ satisfy, $d(Tfx, Tfy) \leq ad(Tx, Ty)$, for all $x, y \in X$
where $0 \leq a < 1$. If the mapping $T$ and $f$ are continuous and $T$ is injective,
sub-sequentially convergent mapping then $f$ has a fixed point in $X$. Moreover,
if $(T, f)$ is a Banach pair then $T$ and $f$ have a unique common fixed point in
$X$.

**Proof:** The proof of corollary follows by taking $b = c = 0$, in theorem 3.1.

**Corollary 3.3.** (Reich type) Let $(X, d)$ be a complete tvs-cone metric
space and $f : X \to X$ satisfies $d(fx, fy) \leq d(x, y) + bd(x, fx) + cd(y, fy)$, for all
$x, y \in X$, where $a, b, c \geq 0$ with $a + b + c < 1$. If the mapping $f$ is continuous
then $f$ has a unique fixed point in $X$.

**Proof:** The proof of this corollary follows by taking $T = I_X$ in theorem 3.1.

**Corollary 3.4.** (T-Kannan type) Let $(X, d)$ be a complete tvs-cone metric
space and $f : X \to X$ satisfy $d(Tfx, Tfy) \leq b[d(Tx, Tfx) + d(Ty, Tfy)]$, for all $x, y \in X$, where $b \in [0, \frac{1}{2})$. If the mappings $T$ and $f$ are continuous and
$T$ is injective, sub-sequentially convergent mapping then $f$ has a unique fixed
point in $X$. Moreover if $(T, f)$ is a Banach pair then $T$ and $f$ have a unique
common fixed point in $X$. 

Proof: The proof of this corollary follows by taking $b = c, a = 0$ in theorem 3.1.

Example 3.5. Let $E = (C_{[0,1]}, \mathbb{R}), P = \{\varphi \in E : \varphi \geq 0\}, X = [0, 1]$ and $d : X \times X \to E$ is defined by $d(x, y)(t) = |x - y|e^t$ where $e^t \in E$. Define $T, f : X \to X$ such that $fx = \frac{x}{2}, Tx = \frac{x}{3}$, then $d(Tfx, Tfy) = d(\frac{x}{2}, \frac{x}{3}) = \frac{1}{6}|x - y|e^t \leq \frac{1}{3}|x - y|e^t = d(Tx, Ty)$

Let $a = \frac{1}{3}, b = \frac{1}{6}, c = \frac{1}{5}$, then clearly $T$ is injective, sub-sequentially convergent and $(T, f)$ is a Banach pair, hence all the conditions of theorem 3.1 are satisfied, and $x = 0$ is the required unique common fixed point.

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