On the Hamiltonian Bigraphs

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Abstract
In this paper we study and discuss simple Hamiltonian biographs and construct a relation of partition of Hamiltonian biographs into independent cycles under certain conditions, atangible results of partition of $K_{n,n}$ into two independent cycles is achieved.

Basic definitions

In this sequel we will introduce basic definitions and concepts of graph theory Which will be used in this paper, all definitions in this paper from (Harary, 1994)

Definition 1.1: A graph $G=(V,E)$ consist of a finite nonempty set of vertices, together with a prescribed set of edges.

Definition 1.2: The number of vertices in $G$ is called order of $G$. denoted by $|V(G)|$.

Definition 1.3: If a graph $G$ is undirected without loops and parallel edges it called simple graph.

Definition 1.4: A walk with distinct edges and distinct vertices is called a path.

Definition 1.5: A cycles is closed path

Definition 1.6: The length of a cycle equal number of edges in it.

Definition 1.7: A bigraph $G$ is a graph with two disjoint sets of vertices $A$ and $B$, with $|V(A)|=m|V(B)|=n$. 


Definition 1.8: A graph $G$ is connected if there exist at least one path between any two vertices in $G$.

Definition 1.9: A connected graph $G$ is Hamiltonian graph if there is a cycle covers all vertices in $G$.

Definition 1.10: The distance $d(u,v)$ between two vertices $u$ and $v$ in $G$ is the length of the shortest path between $u$ and $v$ in $G$.

Definition 1.11: let $A$ be a subgraph of $G$ and let $v \in V(G)$. Then the distance $d(A,v)=d(v)$ is defined as follows:

$$d(v) = d(A, v) = \begin{cases} \min(d(v, u) \text{ if } v \notin V(A), u \in V(A) \\ 0 \text{ if } v \in V(A) \end{cases}$$

Definition 1.12:A biograph with partition $(A,B)$ is called balance if $|A| = |B|$ i.e $K_n,n$

**Partition of Hamiltonian Biagraph into independent cycles**

In this sequel we will discuss finite simple graph, (Bondy & Chvatal, 1976) illustrate that if $G$ is a bipartite graph $K_n,n$ with bipartition $(A,B)$ and for any $x \in A$, $y \in B$ and $d(x) + d(y) \geq n+1$ Then $G$ is Hamiltonian graph

**Our main target is to prove the following result;**

**Theorem 4.1:**

If $K_n,n$ bipartition into $A$ and $B$ such that for any $x \in A$, $y \in B$ and $d(x) + d(y) \geq n+2$, then for any $(n_1,n_2, n= n_1+n_2)$

$G$ contains two independent cycles of length $2n_1,2n_2$.

**Note: The conditions on theorem 4.1 gives that $G$ is Hamiltoians** (Amar, 1986)

**Notation:**
If $F$ and $H$ are disjoint subgraphs of $G$ and if $u_1,\ldots,u_p$ are vertices of $G$, not in $H$, then $H + (u_1,\ldots,u_p)$ is the subgraph of $G$ with vertices $u_1,\ldots,u_p$ and the vertices of $H$.

If $v_i,\ldots,v_k$ are vertices of $H$, $H-(v_i,\ldots,v_k)$ is the subgraph of $G$ with vertices in $H$ except $v_i,\ldots,v_k$. If $C$ is a path or a cycle of $G$, then for any arbitrary orientation; if $u$ is a vertex of $C$, $u^+$ (resp. $u^-$) is the successor (resp. the predecessor) of $u$ on the path or the cycle for the given orientation.

**Remark:**

If $n$ is odd, the result of the theorem is the best one as we can see with the following example: $n=2p+1$. There is no partition into two cycles of lengths $2p$ and $2p+2$ (see e.g. sceam.1).

```
  x   0   x   0
  x
  x
```

**sceam.1**

It is clear that every vertex in $G$ in sceam.1 of degree 4, so $d(x)+d(y)=8$ and $|V(G)|=14$ but $G$ does not contain two independent cycles of lengths 6, 8.

The proof of Theorem is based on many elementary lemmas that we will give first.

**Note: the proof of the following lemmas in thesis”The Cycles of simple Graph”ALrawajfeh,alaa.2012**

**Elementary Lemmas1.1:**
Let $G$ be a balanced bipartite graph with bipartition $(A, B)$, such that $|A| = |B| = n$.

**Lemma 1.1:**

If for any $x \in A, y \in B, d(x, G) + d(y, G) \geq n + 1$, then $G$ is Hamiltonian.

**Lemma 2.1:**

If $G$ contains a Hamiltonian path with endvertices $a$ and $b$ such that $d(a, G) + d(b, G) \geq n + 1$, then $G$ is Hamiltonian.

**Lemma 3.1:**

If there is a partition of $G$ into two paths with endvertices $(a_1, b_1)$ and $(a_2, b_2), a_i \in A, b_i \in B$ such that,

$$d(a_1, G) + d(b_2, G) \geq n + 1,$$

$$d(a_2, G) + d(b_1, G) \geq n + 1,$$

then $G$ is Hamiltonian.

**Lemma 4.1:**

If $\Gamma$ is a path (a cycle) in $G$ with $2p$ vertices and if $(a, b)$ is an edge of $G$ with no vertex in $\Gamma$, such that $d(a, \Gamma) + d(b, \Gamma) \geq p + 1$; then the subgraph $\Gamma + (a, b)$ is traceable (Hamiltonian).
Lemma 5.1:

If $\Gamma$ is a path (a cycle) with $2p$ vertices and if $a \in A$, $b \in B$ are two vertices not in $\Gamma$, such that $d(a, \Gamma) + d(b, \Gamma) \geq p + 2$, then the subgraph $\Gamma + (a, b)$ is traceable (Hamiltonian).

Lemma 6.1:

If $a \in A, b \in B$ are vertices of a cycle $\Gamma$ with $2p$ vertices; such that;

$$d(a^+, \Gamma) + d(b^+, \Gamma) \geq p + 2,$$

$\Gamma$ contains a path $P$ with endvertices $a$ and $b$, such that $V(\Gamma) = V(P)$.

Structure Lemma 2.1:

Structure lemma: if $n = n_1 + n_2$ and for any $x \in A, y \in B$,$d(x, G) + d(y, G) \geq n + 2$, there is a partition of $G$ into two balanced bipartite subgraphs $(G_1, G_2)$ or $(\Gamma_1, \Gamma_2)$ such that one of the following conditions is satisfied:

1. $|V(G_i)| = 2n_i$ and if $x \in A, y \in B$ are in $G_i$, $d(x, G_i) + d(y, G_i) \geq n_i + 2$.
2. $|\Gamma_i| = 2(n_i - 1), \Gamma_i$ is traceable, $|\Gamma_j| = 2(n_j + 1)$, $j \neq i, \Gamma_j$ is Hamiltonian and if $u \in A, v \in B$ are in $\Gamma_j$, $d(u, \Gamma_j) + d(v, \Gamma_j) \geq n_j + 2$.

To prove the theorem, we need to know the structure of $\Gamma_2$ when $\Gamma_2 - (x_0, y_0)$ is not Hamiltonian.
Structure of $\Gamma_2$ when $\Gamma_2-(x_0, y_0)$ is not Hamiltonian:

**Case A.** there is $k$, $2 \leq k \leq n_2 - 1$, such that the edges $(x_1, y_{k+1})$ and $(x_k, y_{n_2})$ exist.

**Lemma 1.2:**

If $\Gamma_2-(x_0, y_0)$ is not Hamiltonian and if the edges $(x_1, y_{k+1})$ and $(x_k, y_{n_2})$ exist, then $x_0$ is adjacent to $y_k$ and $y_0$ is adjacent to $x_{k+1}$ and one of the subgraphs $\Gamma_2-(x_0, y_2)$ or $\Gamma_2-(x_1, y_0)$ is Hamiltonian.

**Case B.** $x_1$ is adjacent to $y_1, y_2, \ldots, y_p$ and $y_{n_2}$ is adjacent to $x_{p+1}, \ldots, x_{n_2}$ for $1 \leq p \leq n_2$.

**Lemma 2.2:**

If $\Gamma_2-(x_0, y_0)$ is not Hamiltonian and if $x_1$ is adjacent to $y_1, \ldots, y_p$ and $y_{n_2}$ is adjacent to $x_{p+1}, \ldots, x_{n_2}$ then

(i) $\{x_1, x_2, \ldots, x_p, y_{p+1}, y_{p+1}, \ldots, y_{n_2}\}$ is an independent set.

(ii) The subgraphs $(x_1, \ldots, x_p)(y_0, y_1, \ldots, y_p)$

$$((y_{p+1}, \ldots, y_{n_2})(x_{p+1}, \ldots, x_{n_2}, x_0))$$

Are complete bipartite subgraphs.
Lemma 3.2:

In case B, if $\Gamma_2 -(x_i, y_j)$ is not Hamiltonian, if $n_2$ is odd, $\Gamma_2$ is the graph $E_1$ with $p = (n_2 + 1)/2$, if $n_2$ is even, $\Gamma_2$ is the graph $E_2$ with $p = (n_2/2)$. For $1 \leq i \leq p - 1$ and $p + 1 \leq j \leq n_2$, the subgraphs $\Gamma_2 -(x_0, y_j)$ and $\Gamma_2 -(x_i, y_j)$ are Hamiltonian.

Proof of the theorem: First Case:

There are two adjacent vertices of $\Gamma_2$, adjacent to the endvertices a and b Hamiltonian path of $\Gamma_1$.

Let $x \in A, y \in B$ be adjacent vertices of $\Gamma_2$ adjacent to b and a. on a cycle of $\Gamma_2$ we consider $x^+, x^-, y^+ , y^-$.

If $x^+, x^-, y^+, y^-$ are not adjacent to $\Gamma_1$;

$$d(x^+, \Gamma_2) + d(y^+, \Gamma_2) \geq n_1 + n_2 + 2,$$

$$d(x^-, \Gamma_2) + d(y^-, \Gamma_2) \geq n_1 + n_2 + 2$$ since we have

$$d(x^-, G) + d(y^-, G) \geq n_1 + n_2 + 2 \& d(x^+, G) + d(y^+, G) \geq n_1 + n_2 + 2.$$

We have $\Gamma_2$ in Hamiltonian and $xy \in E$; and

$$d(x^+, \Gamma_2 -(x, y)) + d(y^+, \Gamma_2 -(x, y)) \geq n_1 + n_2 > n_2 + 1, n_1 \geq 2$$

$$d(x^-, \Gamma_2 -(x, y)) + d(y^-, \Gamma_2 -(x, y)) \geq n_1 + n_2 > n_2 + 1, n_1 \geq 2$$
Hence, there is a partition of $\Gamma_2 -(x, y)$ into two paths with endvertices $(y^+, x^-)$ and $(y^-, x^+)$. (See schem. 12). Thus by Lemma, we have $\Gamma_2 -(x, y)$ is Hamiltonian, and $\Gamma_1 +(x, y)$ and $\Gamma_2 -(x, y)$ are solutions of the problem.

Else, let $(x_0, y_0, x_1, y_1, \ldots x_n, y_n)$ be a Hamiltonian cycle of $\Gamma_2$ such that the chordinality of the pairs of consecutive adjacent to a and b is minimum and suppose $(x_0, y_0)$ be adjacent to b and a. If $\Gamma_2 -(x_0, y_0)$ is Hamiltonian, $\Gamma_1 +(x_0, y_0)$ and $\Gamma_2 -(x_0, y_0)$ are the solutions of the problem.

Suppose $\Gamma_2 -(x_0, y_0)$ is not Hamiltonian. We consider case A and Case B the presented paragraph.

![schem.10](image)

**Case A:**

By Lemma $x_0$ is adjacent to $y_k$ and $y_0$ is adjacent to $y_{k+1}$, we consider the Hamiltonian cycle of $\Gamma_2 : (y_0, x_1, x_k, y_k, x_0, y_2, \ldots x_{k+1}, y_0)$. By hypothesis of minimality, one of edges at least (a, $y_k$) or (b, $x_{k+1}$) exists, and $\Gamma_1 +(x_0, y_k)$ and $\Gamma_2 -(x_0, y_k)$ or $\Gamma_1 +(y_0, x_{k+1})$ and $\Gamma_2 +(y_0, x_{k+1})$ are the solutions of the problem; since $\Gamma_1 +(x_0, y_k)$ has a Hamiltonian cycle say; $y_k a \ldots b x_0 y_k$, also, $\Gamma_2 -(x_0, y_k)$ has a Hamiltonian cycle:

$$y_0 x_1 y_1 \ldots x_k y_n x_{n+2} \ldots y_{k+1} x_{k+1} x_0$$
note that in case \((x_1, y_{k+1}) \& (y_n, x_k)\) are edges in \(\Gamma_2\). Similarly; if the other case hold.

**Case B. Subcase I:**

The endvertices a and b of a Hamiltonian path of \(\Gamma_1\) are adjacent to three consecutive vertices of Hamiltonian cycle of \(\Gamma_2\).

We can suppose that a is adjacent to \(y_0\) and b is adjacent to \(x_0\) and \(x_1\).

If \(\Gamma_2 - (x, y)\) is a Hamiltonian, \(\Gamma_1 + (x, y_0)\) and \(\Gamma_2 - (x, y_0)\) are solutions of the problem. \((y_0a \ldots b_{(x_1y_0)}\) is Hamiltonian cycle of \(\Gamma_1 + (x, y_0)\)). Else \(\Gamma_2 - (x, y_0)\) is not Hamiltonian, by Lemma, \(\Gamma_2\) is the graph \(E_1\) if \(n_2\) is odd, the graph \(E_2\) if \(n_2\) is even and the subgraphs \(\Gamma_2 - (x, y)\) for \(p+1 \leq j \leq n_2\) or \(\Gamma_2 - (x, y)\) for \(1 \leq i \leq p - 1\) are Hamiltonian.

If a is not adjacent to \(y_i\) \(1 \leq i \leq p - 1\), or \(y_i\) \(p + 1 \leq j \leq n_2\) \(d(a, \Gamma_2) \leq 2\). By lemma, for \(p + 1 \leq j \leq n_2\) \(d(y_j, \Gamma_2) = n_2 + 1 - p\) (Since \(y_i\) is independent with vertices \(x_1, x_2, \ldots, x_p\), \(p + 1 \leq j \leq n_2\), then

\[
d(a, \Gamma_2) + d(y_j, \Gamma_2) \leq n_2 + 3 - p;
\]

\[
d(a, \Gamma_2) + d(y_j, \Gamma_1) \geq n + 2 - (n_2 + 3 - p) = n_1 + p - 1;
\]

\(\Gamma_1 + (x_0, y_i)\) contains a Hamiltonian path with endvertices a, \(y_i\) say \((a, \ldots, b, x_0, y_i)\). By previous Lemma, \(\Gamma_1 + (x_0, y_i)\) is Hamiltonian (since \(d(a, \Gamma_1^*) + d(y_j, \Gamma_1^*) \geq n_1 + p \geq n_1 + 1\), where \(\Gamma_1^* = \Gamma_1 + (x_0, y_i)\)).

Hence, \(\Gamma_1 + (x_0, y_i)\) and \(\Gamma_2 - (x_0, y_i)\) are solutions of problem.
Subcase II:

The endvertices $a$ and $b$ of a Hamiltonian path of $\Gamma_1$ are adjacent to $y_k$ and $x_k$, $k \neq 0$.

If $1 \leq k \leq p - 1$ or $p + 2 \leq k \leq n_2$, $\Gamma_1 + (x_k, y_k)$ and $\Gamma_2 - (x_k, y_k)$ are the solution of the problem.

If $k = p$, the edes $(x_1, y_p)$ and $(x_0, y_p)$ exist; the vertices $x_0, y_0$ and $x_p$ are consecutive on the Hamiltonian cycle $(x_0, y_0, x_p, y_{p-1}, x_1, y_{p-1}, x_{p+1}, x_2, y_{p+1}, x_0)$ and we can conclude as in the first subcase.

If $k = p + 1$, the edes $(x_0, y_{p+1})$ and $(y_{n_2}, x_{p+1})$ exist; as a similar argument, the vertices $x_0, y_0, y_{p+1}$ are consecutive on the Hamiltonian cycle $(y_0, x_0, y_{p+1}, x_1, y_{n_2}, x_{p+1}, x_2, y_2)$ and we can conclude as in the first subcase.

Subcase III:

The endvertices $a$ and $b$ of a Hamiltonian path of $\Gamma_1$ satisfy $d(a, \Gamma_2) + d(b, \Gamma_2) = n_2 + 2$.

If $1 \leq k \leq p - 1$ or $p + 1 \leq k \leq n_2 - 1$, $\Gamma_1 + (x_k, y_{k+1})$ and $\Gamma_2 - (y_k, x_{k+1})$ are solution of the problem. Note that $\Gamma_2 - (y_k, x_{k+1})$ has Hamiltonian path with endvertices $x_k$ and $y_{k+1}$. 
If $k = p$, $p \leq n_2 - 2$, $p + 2 \leq j \leq n_2$, $d(y_j, \Gamma_2) = n_2 - p + 1$ ($y_j$ is adjacent to $(x_{p+1}, x_{p+2}, \ldots, x_{n_2}, x_0)$; $d(a, \Gamma_2) = p + 1$; since $a$ is adjacent to $y_1, \ldots, y_p, y_0$, then $d(a, \Gamma_1) + d(y_j, \Gamma_1) \geq d(a, G) + d(y_j, G) - d(a, \Gamma_2) - d(y_j, \Gamma_2)$
\[\geq n_1 + n_2 + 2 - (n_2 - p + 1) - (p + 1)\;.

As $d(a, \Gamma_1) + d(y_j, \Gamma_1) \geq n_i$ and $|\Gamma_1| = 2(n_i - 1)$, then there is an edge $uv$ of $\Gamma_1$ with $u \in A$ and $\{av, y_ju\} \subseteq E$; see sceam.10, and we have $bx_j \in E$, for $p + 2 \leq j \leq n_2$, so $\Gamma_1 + (x_j, y_j)$ has a Hamiltonian cycle namely $a, \ldots, uy_jx_jb, \ldots, v, a$. Hence $\Gamma_1 + (x_j, y_j)$ is Hamiltonian. Not that $\Gamma_2 - (x_j, y_j)$ has a Hamiltonian path with endvertices $y_{j-1}$ and $x_{j+1}$ and $\{x_{j+1}y_{n_2}, y_{j-1}x_{n_2}\} \subseteq E$.

Thus $\Gamma_1 + (x_j, y_j)$ and $\Gamma_2 - (x_j, y_j)$ are solution of the problem.

The case $p = n_2 - 1, p \geq 2$ is similar, and if $p = n_2 - 1 = 1$, so $n_2 = 2, p = 1$, therefore, $a$ is adjacent to $y_1, y_0$ and $b$ is adjacent to $x_1, x_0$, then $d(a, \Gamma_1) + d(b, \Gamma_1) \geq n_1$ so $\Gamma_1$ is Hamiltonian by Lemma.

**Subcase IV:**

For any Hamiltonian path of $\Gamma_1$, the endvertices $\alpha$ and $\beta$ satisfy $d(\alpha, \Gamma_2) + d(\alpha, \Gamma_2) \geq n_2 + 1$.

**Lemma 3.2:**

Under the hypothesis of subcase IV, $\Gamma_1$ is Hamiltonian and if $\alpha \in A$ and $\beta \in B$ are in $\Gamma_1$, then
\[ d(\alpha, \Gamma_1) + d(\beta, \Gamma_1) \geq n_1 + 1, \]
\[ d(\alpha, \Gamma_2) + d(\beta, \Gamma_2) \leq n_2 + 1, \]

And there is a Hamiltonian path in \( \Gamma_1 \) with endvertices \( \alpha \) and \( \beta \).

**Proof:** let \( \alpha \) and \( \beta \) be the endvertices of a Hamiltonian path of \( \Gamma_1 \):
\[ d(\alpha, \Gamma_1) + d(\beta, \Gamma_1) \geq n_1 + 1. \]

By Lemma 2.1 \( \Gamma_1 \) is Hamiltonian, if \( \alpha^+ \) is the successor of \( \alpha \), on a Hamiltonian cycle of \( \Gamma_1 : d(\alpha, \Gamma_1) + d(\alpha^+, \Gamma_1) \geq n_1 + 1 \) (\( \alpha, \alpha^+ \) are endvertices of a Hamiltonian path of \( \Gamma_1 \)).

Suppose that \( u \in A \) and \( v \in B \) are in \( \Gamma_1 \), and satisfy \( d(u, \Gamma_1) + d(v, \Gamma_1) \leq n_i \), then
\[ d(u, \Gamma_1) + d(u^+, \Gamma_1) \geq n_1 + 1; \]
\[ d(v, \Gamma_1) + d(v^+, \Gamma_1) \geq n_1 + 1; \]

Implies that, \( d(u^+, \Gamma_1) + d(v^+, \Gamma_1) \geq n_1 + 2 \). Therefore by Lemma \( \Gamma_1 \) contains a Hamiltonian path with endvertices \( u, v \), that contradicts our hypothesis.

**Proof of the theorem in subcase IV:**
\[ d(x_1, \Gamma_2) + d(y_{n_2}, \Gamma_2) = n_2 + 2, \] so \( d(x_1, \Gamma_1) + d(y_{n_2}, \Gamma_1) \geq n_1 \), thus \( x_1 \) and \( y_{n_2} \) are adjacent to \( \Gamma_1 \). By lemma 4.8.3 one of the subgraphs \( \Gamma_2 - (x_1, y_0) \) or \( \Gamma_2 - (x_0, y_{n_2}) \) is Hamiltonian.

If \( \Gamma_2 - (x_1, y_0) \) is a Hamiltonian. Let \( \delta \in \Gamma_1 \) be adjacent to \( x_1 \). By Lemma;
\[ d(\delta^+, \Gamma_1) + d(\alpha^+, \Gamma_1) \geq n_1 + 1. \]
By Lemma $\Gamma_1$ contains a Hamiltonian path with endvertices $\delta$, a. Hence $\Gamma_1 + (x_1, y_0)$ is Hamiltonian, since $(y_0 a, \ldots, \delta x_1 y_0)$ is a Hamiltonian cycle of $\Gamma_1 + (x_1, y_0)$. Hence $\Gamma_1 + (x_1, y_0)$ and $\Gamma_2 - (x_1, y_0)$ are solutions of the problem.

**Proof of the theorem: Second Case:**

For any Hamiltonian path of $\Gamma_1$, its endvertices $a$ and $b$ are not adjacent to two adjacent vertices of $\Gamma_2$.

**Lemma 3.3:**

Under the hypothesis of second case, if $\alpha \in A, \beta \in B$ are in $\Gamma_1$.

\[
d(\alpha, \Gamma_2) \geq 2, \quad d(\beta, \Gamma_2) \geq 2
\]

\[
d(\alpha, \Gamma_1) \geq 2, \quad d(\beta, \Gamma_1) \geq n_1 + 2,
\]

\[
d(\alpha, \Gamma_2) + d(\beta, \Gamma_2) \leq n_2
\]

**Proof:** Let $u \in A$ in $\Gamma_2$ be not adjacent to $\Gamma_1$ and $b \in \beta$ in $\Gamma_1$,

\[
d(b, \Gamma_2) \geq n + 2 - d(u, G) - d(b, \Gamma_1), \text{ since } d(u, G) + d(b, G) \geq n_1 + n_2 + 2;
\]

\[
d(b, G) = d(b, \Gamma_1) + d(b, \Gamma_2);
\]

\[
d(u, G) = d(u, \Gamma_2)
\]

$u$ is not adjacent to $\Gamma_1$; and $d(u, G) \leq n_2 + 1, d(b, \Gamma_1) \leq n_1 - 1$.

Therefore,

\[
d(b, \Gamma_2) \geq n_1 + n_2 + 2 - n_2 - 1 - n_1 + 1\]

\[
d(b, \Gamma_2) \geq 2;
\]

$b$ is adjacent to vertices $x \in A$ in $\Gamma_2$. Then $y = x^+$ is not adjacent to $\Gamma_1$ (since any two endvertices of $\Gamma_1$ is not adjacent to two adjacent vertices of $\Gamma_2$).
By a similar argument if \( a \in A \) is in \( \Gamma_1 \), \( x' \in \Gamma_2 \), \( x' \) is not adjacent to \( \Gamma_1 \), \( d(a, \Gamma_2) \geq 2 \).

Suppose there are \( \alpha \in A, \ B \in \beta \), two vertices of \( \Gamma_1 \) that satisfy
\[
d(\alpha, \Gamma_1) + d(\beta, \Gamma_1) = n_1 + 1
\]
So \( d(\alpha, \Gamma_2) + d(\beta, \Gamma_2) = n_2 + n_1 + 2 - n_1 - 1 = n_2 + 1 \).

Necessarily \( \alpha \) and \( \beta \) are adjacent to two adjacent vertices of \( \Gamma_2 \) (Since \( |\Gamma_2| = 2(n_2 + 1) \) and \( d(\alpha, \Gamma_2) + d(\beta, \Gamma_2) = n_2 + 1 \)), which contradicts our hypothesis:

i.e., \( d(\alpha, \Gamma_1) + d(\beta, \Gamma_1) \geq n_1 + 2 \)

so \( d(\alpha, \Gamma_2) + d(\beta, \Gamma_2) \leq n_2 \)

**proof of the theorem in the second case:**

Let \( a \in A \) and \( b \in \beta \) be two vertices of \( \Gamma_1 \), adjacent to \( y \in B \) and \( x \in A \) in \( \Gamma_2 \). \( x \) and \( y \) are adjacent to two vertices, consecutive on a Hamiltonian cycle of \( \Gamma_2 \), \( y' \in B \) and \( x' \in A \) (note that, \( x'y' \in E \) and \( d(x, \Gamma_2) + d(y, \Gamma_2) \geq n_2 + 2 \).

Let \( \Gamma_1^- \) is obviously Hamiltonian, since \( x'ya,...,bxy'x' \) is Hamiltonian cycle of \( \Gamma_1^- \).

Let \( u \in A \) and \( v \in B \) be two vertices of \( \Gamma_2^- \). We distinguish three cases:

(i) \( u \) and \( v \) are not adjacent to \( \Gamma_1 \).

(ii) \( u \) is adjacent to \( \Gamma_1 \) and then \( u \) is not adjacent to \( y \) (since \( y \) is adjacent to \( \Gamma_1 \)).
(iii) $u$ and $v$ are adjacent to $\Gamma_1$ and there are not adjacent to $y$ and $x$.

In each case we can conclude that:

$$d(u, \Gamma_2^+ ) + d(v, \Gamma_2^+) \geq n_2;$$

And by Lemma 1.1, $\Gamma_2^+$ is Hamiltonian ($|\Gamma_2^-| = 2(n_2 - 1)$).

Let $(ab\alpha_2\beta_2...\alpha_{n-1}\beta_{n-1})$ be a Hamiltonian cycle of $\Gamma_1$ and; if $n_i \geq 5$, let for $3 \leq i \leq n_i - 2, \alpha_i = \alpha, \beta_i = \beta$ be two vertices of $\Gamma_1$ different from $a$ and $b$. $\alpha$ and $\beta$ are adjacent to $\Gamma_2^-$ in $y_1 \in B$ and $x_i \in A(x_i, y_1 \notin E)$. If $x_i^+$ and $y_i^+$ are the successors of $x_i$ and $y_i$ on Hamiltonian cycle of $\Gamma_2^-$.

$$d(y_i^+, \Gamma_2^-) + d(x_i^+, \Gamma_2^-) \geq n_i + n_2 + 2;$$

(since $x_i^+, y_i^+$ are not adjacent to $\Gamma_1$), then

$$d(y_i^+, \Gamma_2^-) + d(x_i^+, \Gamma_2^-) \geq n_i + n_2 - 2, |\Gamma_2^-| = 2(n_2 - 1) \geq n_2 + 3$$

By Lemma 4.2.6 $\Gamma_2^-$ contains a Hamiltonian path with endvertices $x_1$, $y_1$ respectively; hence $\Gamma_2^- + (\alpha, \beta)$ is Hamiltonian $(\alpha, y_1, ..., x_i, \beta, \alpha_i)$ is Hamiltonian cycle of $\Gamma_2^- + (\alpha, \beta)$.

Let $\alpha^- = \beta_{i-1}, \beta^+ = \alpha_{i+1}$. By Lemma

$$d(\alpha^-, \Gamma_1) + d(\beta^+, \Gamma_1) \geq n_i + 2.$$ 

We can deduce that $\Gamma_i^-(\alpha, \beta)$ is Hamiltonian to illustrate this;
\[ \Gamma_i^- \text{ is Hamiltonian i.e. } x' ya, ..., bxy' x' \text{ is Hamiltonian cycle of } \Gamma_i^-, \]
\[ \alpha = \alpha_i, \beta = \beta_i \text{ and } \Gamma_i^- - (\alpha, \beta) \text{ is Hamiltonian path with endvertices } \alpha^-, \beta^+ \text{ and since} \]
\[ d(\alpha^-, \Gamma_i^-) + d(\beta^+, \Gamma_i^-) \geq n_i + 2 \text{ so,} \]
\[ d(\alpha^-, \Gamma_i^-, (\alpha, \beta)) + d(\beta^+, \Gamma_i^-, (\alpha, \beta)) \geq n_i + 1, |\Gamma_i^- - (\alpha, \beta)| = 2n_i \]

By Lemma 4.2.2 \( \Gamma_i^- - (\alpha, \beta) \) is Hamiltonian. (This case satisfied if at least one of the edges \( \alpha^-y \) or \( \beta^+x \) exist).

If \( n_i \geq 5, \Gamma_i^- - (\alpha, \beta) \) and \( \Gamma_2^- - (\alpha, \beta) \) are solutions of the problem.

If \( n_i \leq 4 \), it’s easy case; to see this argument, if \( n_i-3 \), then \( (ab\alpha_i\beta_i) \) is Hamiltonian cycle of \( \Gamma_i \), let \( \alpha_i, \beta_i \) be adjacent to \( x_i \in A, y_i \in B \) in \( \Gamma_2^- \). Then \( \Gamma_2^- + (\alpha_i, \beta_i) \) is Hamiltonian (by the previous argument) and it’s obviously \( \Gamma_i^- - (\alpha_i, \beta_i) \) is Hamiltonian \( (abxy' x' y) \) is Hamiltonian cycle of \( \Gamma_i^- - (\alpha_i, \beta_i) \), (See sceam.11), following the same argument for \( n_i=4 \).

This completes the proof of the theorem.
References


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