Jensen and Hermite-Hadamard inequalities for strongly convex set-valued maps

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Abstract

Counterparts of the classical integral and discrete Jensen inequalities and the Hermite-Hadamard inequalities for strongly convex set-valued maps are presented.

Mathematics Subject Classification: Primary 26A51. Secondary 39B62, 54C60

Keywords: Convex set-valued map, strongly convex set-valued map, Jensen inequality, Hermite-Hadamard inequality
1 Introduction

Let $I \subset \mathbb{R}$ be an interval and $c$ be a positive number. Following Polyak [16] a function $f: I \to \mathbb{R}$ is called strongly convex with modulus $c$ if
\[ f(tx_1 + (1-t)x_2) \leq tf(x_1) + (1-t)f(x_2) - ct(1-t)(x_1 - x_2)^2 \] (1)
for all $x_1, x_2 \in I$ and $t \in [0,1]$. $f$ is called strongly concave with modulus $c$ if $-f$ is strongly convex with modulus $c$. Many properties and applications of strongly convex functions can be found in the literature (see, for instance, [9], [12], [17], [15], [22]). Recently Huang [5], extended the definition (1) of strongly convex function to set-valued maps. He used such maps to investigate error bounds for some inclusion problems with set constraints. Some further properties of strongly convex set-valued maps can be found in [6]. Strongly concave set-valued maps were investigated in [8].

The aim of this paper is to present counterparts of the integral and discrete Jensen inequalities and the Hermite-Hadamard double inequalities for strongly convex set-valued maps.

2 Preliminaries

Throughout this paper $Y$ be a Banach space, $B$ be a closed unit ball in $Y$, $I \subset \mathbb{R}$ be an open interval and $c$ be a positive constant.

Denote by $n(Y)$ the family all nonempty subsets of $Y$ and by $cl(Y)$ the family of all closed nonempty subsets of $Y$. A set-valued map $F: I \to n(Y)$ is called strongly convex with modulus $c$ if
\[ tF(x_1) + (1-t)F(x_2) + ct(1-t)(x_1 - x_2)^2B \subset F(tx_1 + (1-t)x_2) \] (2)
for all $x_1, x_2 \in I$ and $t \in [0,1]$ (see [5], [6]). The usual notion of convex set-valued maps corresponds to relation (2) with $c = 0$ (cf. e.g. [2], [3], [11], [20], [21]).

Clearly, the definition of strongly convex set-valued maps is motivated by that of strongly convex functions. The following lemma characterizes strongly convex set-valued maps with values in $cl(\mathbb{R})$ and shows connections between conditions (1) and (2) (cf. [7] where analogous result for convex set-valued maps is given).

**Lemma 2.1** A set-valued map $F: I \to cl(\mathbb{R})$ is strongly convex with modulus $c$ if and only if it has one of the following forms:

a) $F(x) = [f_1(x), f_2(x)], \ x \in I$,
b) $F(x) = [f_1(x), +\infty), \ x \in I$,
c) $F(x) = (-\infty, f_2(x)], \ x \in I$,,
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\( F(x) = (-\infty, +\infty), \quad x \in I, \)

where \( f_1 : I \to \mathbb{R} \) is strongly convex with modulus \( c \) and \( f_2 : I \to \mathbb{R} \) is strongly concave with modulus \( c \).

Proof. The “if” part is clear. To prove the “only if” part note first that by (2) the values of \( F \) are convex. Moreover, if \( F(x_0) \) is bounded from above (from below) for some \( x_0 \in I \), then \( F(x) \) is bounded from above (from below) for every \( x \in I \). Define

\[
  f_1(x) = \inf F(x), \quad \text{if } F(x) \text{ is bounded from below}
\]

and

\[
  f_2(x) = \sup F(x), \quad \text{if } F(x) \text{ is bounded from above}.
\]

Then by the strong convexity of \( F \) its follows that \( f_1 \) is strongly convex with modulus \( c \) and \( f_2 \) is strongly concave with modulus \( c \). Since the values of \( F \) are closed and convex, the result follows. \( \square \)

3 The Jensen inequalities

It is well know that if a function \( f : I \to \mathbb{R} \) is convex, then if satisfies the integral Jensen inequalities

\[
f \left( \int_X \varphi(x) d\mu \right) \leq \int_X f(\varphi(x)) d\mu \tag{3}\]

for each probability measure space \((X, \Sigma, \mu)\) and all \( \mu \)-integrable functions \( \varphi : X \to I \).

In [9] the following version of the Jensen inequality for strongly convex functions was proved:

\[
f \left( \int_X \varphi(x) d\mu \right) \leq \int_X f(\varphi(x)) d\mu - c \int_X (\varphi(x) - m)^2 d\mu \tag{4}\]

where \( m = \int_X \varphi(x) d\mu \). A counterpart of (3) for set-valued maps was obtained in [7]. The next Theorem gives a counterpart of (4) for set-valued maps.

Throughout this paper the integral of a set-valued map is understood in the sense of Aumann, i.e. it is the set of integrals of all integrable selections of this map.

**Theorem 3.1** Let \((X, \Sigma, \mu)\) be a probability measure space. If \( F : I \to \text{cl}(Y) \) is strongly convex with modulus \( c \), then for each square-integrable function \( \varphi : X \to I \)

\[
\int_X F(\varphi(x)) d\mu + c \int_X (\varphi(x) - m)^2 d\mu \subset F \left( \int_X \varphi(x) d\mu \right), \tag{5}\]

where \( m = \int_X \varphi(x) d\mu \).
Proof. The proof is divided into two steps. First, we assume that $Y = \mathbb{R}$. Then, by Lemma 2.1, $F$ has one of the forms a)- d). Assume that $F(x) = [f_1(x), f_2(x)], x \in I$ (the proof in the remaining cases is similar). Let $h : X \rightarrow \mathbb{R}$ be a $\mu$-integrable selection of $F \circ \varphi$. Then, by the Jensen inequality for strongly convex function (4), we have

$$f_1 \left( \int_X \varphi(x) d\mu \right) \leq \int_X f_1(\varphi(x)) d\mu - c \int_X (\varphi(x) - m)^2 d\mu$$

and

$$f_2 \left( \int_X \varphi(x) d\mu \right) \geq \int_X f_2(\varphi(x)) d\mu + c \int_X (\varphi(x) - m)^2 d\mu$$

Hence

$$\int_X (h(x)) d\mu + c \int_X (\varphi(x) - m)^2 d\mu [-1, 1] \subset F \left( \int_X \varphi(x) d\mu \right).$$

Consequently

$$\int_X F(\varphi(x)) d\mu + c \int_X (\varphi(x) - m)^2 d\mu [-1, 1] \subset F \left( \int_X \varphi(x) d\mu \right),$$

which finishes the proof in the case $Y = \mathbb{R}$.

Now, assume that $Y$ is an arbitrary Banach space. Take a nonzero continuous linear functional $y^* \in Y^*$ and consider the set-valued map $x \mapsto y^*(F(x))$, $x \in I$. This set-valued map is strongly convex with modulus $c||y^*||$ and has closed values in $\mathbb{R}$. Therefore, by the previous step,

$$\int_X y^*(F(\varphi(x))) d\mu + c||y^*|| \int_X (\varphi(x) - m)^2 d\mu [-1, 1] \subset y^* \left( F \left( \int_X \varphi(x) d\mu \right) \right). \quad (6)$$

Fix a point $b \in B$ and take an arbitrary $\mu$-integrable selection $h$ of $F \circ \varphi$. Then, by (6) and the fact that

$$\int_X y^*(h(x)) d\mu = y^* \left( \int_X h(x) d\mu \right),$$

we get

$$y^* \left( \int_X h(x) d\mu + c \int_X (\varphi(x) - m)^2 d\mu b \right)$$

$$\in \int_X y^*(h(x)) d\mu + c||y^*|| \int_X (\varphi(x) - m)^2 d\mu [-1, 1]$$

$$\subset y^* \left( F \left( \int_X \varphi(x) d\mu \right) \right).$$
Since this condition holds for arbitrary \( y^* \in Y^* \) and the set \( y^*(\int X (\varphi(x)d\mu)) \) is convex closed, by the separation theorem (see [18], Corollary 2.5.11) we obtain
\[
\int_X h(x)d\mu + c \int_X (\varphi(x) - m)^2 d\mu b \in F \left( \int_X \varphi(x)d\mu \right)
\]
Thus
\[
\int_X F(\varphi(x))d\mu + c \int_X (\varphi(x) - m)^2 d\mu B \subset F \left( \int_X \varphi(x)d\mu \right),
\]
which was to be proved. \( \square \)

Now, assume that \( X = I \), \( \varphi(x) = x \) for \( x \in I \), and \( x_1, \ldots, x_n \in I \) are distinct points. Moreover, assume that \( \mu \) is a probability measure concentrate at \( x_1, \ldots, x_n \), that is \( \mu(x_1) = t_1 > 0 \), \( i = 1, \ldots, n \) and \( t_1 + \cdots + t_n = 1 \). Then
\[
m = \int_X \varphi(x)d\mu = \sum_{i=1}^n t_i x_i, \quad \int_X (\varphi(x) - m)^2 d\mu = \sum_{i=1}^n t_i (x_i - m)^2
\]
and
\[
\int_X F(\varphi(x))d\mu = \sum_{i=1}^n t_i F(x_i).
\]

Therefore, as the consequence of Theorem 3.1, we get the following discrete Jensen inequality for strongly convex set-valued maps.

**Corollary 3.2** If \( f : I \to \text{cl}(Y) \) is strongly convex with modulus \( c \), then
\[
\sum_{i=1}^n t_i F(x_i) + c \sum_{i=1}^n t_i (x_i - m)^2 B \subset F \left( \sum_{i=1}^n t_i x_i \right)
\]
for all \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in I \), \( t_1, \ldots, t_n > 0 \) with \( t_1 + \cdots + t_n = 1 \) and \( m = t_1 x_1 + \cdots + t_n x_n \).

4 The Hermite-Hadamard inequality

It is known that if a function \( f : I \to \mathbb{R} \) is convex then it satisfies the Hermite-Hadamard double inequality
\[
f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}, \quad a, b \in I, \quad a < b.
\]
The following version of the Hermite-Hadamard inequality for strongly convex functions was recently proved in [9]:

\[ f \left( \frac{a + b}{2} \right) + \frac{c}{12} (a - b)^2 \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} - \frac{c}{6} (a - b)^2, \quad (8) \]

for all \( a, b \in I, \ a < b. \)

In this section we present a counterpart of the above inequality (8) for strongly convex set-valued maps. The Hermite-Hadamard inequality for convex set-valued maps was obtained in [19] (cf. also [14], [10]) .

**Theorem 4.1** If a set-valued map \( F: I \to \text{cl}(Y) \) is strongly convex with modulus \( c \), then

\[
\frac{1}{b-a} \int_a^b F(x) dx + \frac{c}{12} (a - b)^2 B \subset F \left( \frac{a + b}{2} \right) \quad (9)
\]

and

\[
\frac{F(a) + F(b)}{2} + \frac{c}{6} (a - b)^2 B \subset \frac{1}{b-a} \int_a^b F(x) dx \quad (10)
\]

for all \( a, b \in I, \ a < b. \)

**Proof.** Condition (9) follows from Theorem 3.1. To show this take \( X = [a, b], \varphi(x) = x, x \in [a, b] \) and \( \mu = \frac{1}{b-a} \lambda \), where \( \lambda \) is the Lebesgue measure on \( \mathbb{R} \). Then

\[
m = \int_X \varphi(x) d\mu = \frac{a + b}{2}, \quad F \left( \int_X \varphi(x) d\mu \right) = F \left( \frac{a + b}{2} \right),
\]

\[
\int_X (\varphi(x) - m)^2 d\mu = \frac{1}{2} (a - b)^2 \quad \text{and} \quad \int_X F(\varphi(x)) d\mu = \frac{1}{b-a} \int_a^b F(x) dx.
\]

Substituting these equalities to (5) we get (9).

To prove condition (10) take arbitrary \( z = \frac{a + b}{2} + \frac{c}{6} (a - b)^2 \beta \), where \( u \in F(a), v \in F(b) \) and \( \beta \in B \). Considerer the function \( f: [a, b] \to Y \) defined by

\[
f(x) = \frac{b-x}{b-a} u + \frac{x-a}{b-a} v + c(b-x)(x-a) \beta.
\]

By the strong convexity of \( F \) we get

\[
f(x) \in \frac{b-x}{x-a} F(a) + \frac{x-a}{b-a} F(b) + c \frac{b-x}{b-a} \frac{x-a}{b-a} (b-a)^2 B \subset F \left( \frac{b-x}{b-a} a + \frac{x-a}{b-a} b \right) = F(x),
\]
which means that $f$ is a selection of $F$.

Simple calculations gives

$$\int_a^b f(x)dx = (b-a) \left[ \frac{u+v}{2} + \frac{1}{6}c\beta(a-b)^2 \right] = (b-a)z.$$ 

Hence

$$z = \frac{1}{b-a} \int_a^b f(x)dx \in \frac{1}{b-a} \int_a^b F(x)dx,$$

which finishes the proof. \qed

5 The converse of Hermite-Hadamard theorem

It is known that if a continuous function $f : I \to \mathbb{R}$ satisfies the left or the right-hand side inequality in (7), then it is convex (cf. e.g. [2], [4], [13]). An analogous result holds also for strong convexity: If $f : I \to \mathbb{R}$ is continuous and satisfies the left or the right-hand side inequality in (8), then it is strongly convex with modulus $c$ (see [9]). In this section we present a set-valued counterpart of that result. Recall that a set-valued map $F : I \to \mathfrak{n}(Y)$ is said to be continuous at a point $x_0$ if for every neighbourhood $V$ of zero in $Y$ there exist a neighbourhood $U$ of zero in $\mathbb{R}$ such that

$$F(x) \subset F(x_0) + V \quad \text{and} \quad F(x_0) \subset F(x) + V$$

for all $x \in (x_0 + U) \cap I$.

In what follows we assume that $Y$ is a separable Banach space and denote by $bccl(Y)$ the family of all bounded convex closed and non-empty subsets of $Y$.

**Theorem 5.1** If $F : I \to bccl(Y)$ is continuous and satisfies

$$\frac{1}{b-a} \int_a^b F(x)dx + \frac{c}{12}(a-b)^2B \subset F\left(\frac{a+b}{2}\right), \quad a, b \in I, \quad a < b. \quad (11)$$

or

$$\frac{F(a) + F(b)}{2} + \frac{c}{6}(a-b)^2B \subset \frac{1}{b-a} \int_a^b F(x)dx, \quad a, b \in I, \quad a < b, \quad (12)$$

then $F$ is strongly convex with modulus $c$. 
Proof. Assume that $F$ satisfies (11) (if $F$ satisfies (12) the proof is analogous). Define $G(x) = F(x) + cx^2 B$, $x \in I$. Then

\[
\frac{1}{b-a} \int_{a}^{b} G(x) \, dx = \frac{1}{b-a} \int_{a}^{b} F(x) \, dx + \frac{1}{b-a} \int_{a}^{b} cx^2 B \, dx \\
= \frac{1}{b-a} \int_{a}^{b} F(x) \, dx + c \left( \frac{a^2 + ab + b^2}{3} \right) B \\
= \frac{1}{b-a} \int_{a}^{b} F(x) \, dx + c \left( \frac{(a-b)^2}{12} B + c \left( \frac{a+b}{2} \right)^2 B \right) \\
\subset F\left( \frac{a+b}{2} \right) + c \left( \frac{a+b}{2} \right)^2 B = G\left( \frac{a+b}{2} \right).
\]

Thus $G$ satisfies the Hermite-Hadamard-type inclusion and it is also continuous. Therefore, by [10, Theorem 8], $G$ is convex. Hence, using the definition of $G$ and the characterization of strongly convex set-valued maps given in [6], we obtain that $F$ is strongly convex with modulus $c$. This finished the proof.

\[\square\]

References


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Received: November, 2014