Rota Baxter operators of the simple 3-Lie algebra I

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Abstract

This paper studies Rota-Baxter operators on the simple 3-Lie algebras over the complex field. It is proved that there does not exist Rota-Baxter operators of weight zero with rank 3 on the simple 3-Lie algebras. And it provides the Rote-Baxter operators of weight zero with rank 1, 2, 4, respectively.

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1 Introduction

We know that 3-Lie algebras [1] have wide applications in many fields of mathematics and mathematical physics[2]. In recent years, kinds of multiple algebras are studied [3-5]. For example, Rota Baxter 3-Lie algebra was introduced in the paper [5], and the structure of Rota Baxter 3-Lie algebras is discussed. Rota-Baxter (associative) algebras, originated from the work of G. Baxter [6] in probability and populated by the work of Cartier and Rota [7] have connections with many areas of mathematics and physics, including combinatorics,
number theory, operads and quantum field theory. In particular Rota-Baxter algebras have played an important role in the Hopf algebra approach of renormalization of perturbative quantum field theory of Connes and Kreimer [7], as well as in the application of the renormalization method in solving divergent problems in number theory [8].

In this paper we investigate the existence of Rota-Baxter operators of the weight zero on the simple 3-Lie algebras over the complex field. First we introduce some basic notions.

A 3-Lie algebra is a vector space $A$ over a field $F$ endowed with a 3-ary multi-linear skew-symmetric operation $[ , , ]$ satisfying the 3-Jacobi identity, $\forall x_1, x_2, x_3, y_2, y_3 \in A$,

$$[[x_1, x_2, x_3], y_2, y_3] = \sum_{i=1}^{3} [x_1, \cdots, [x_i, y_2, y_3], \cdots, x_3], \forall x_1, x_2, x_3 \in L. \quad (1)$$

Let $A$ be a 3-Lie algebra, $\lambda \in F$, if a linear mapping $P : A \rightarrow A$ satisfies

$$[P(x_1), P(x_2), P(x_3)] = P([P(x_1), P(x_2), x_3] + [P(x_1), x_2, P(x_3)]$$

$$+ [x_1, P(x_2), P(x_3)] + \lambda [P(x_1), x_2, x_3] + \lambda [x_1, P(x_2), x_3]$$

$$+ \lambda [x_1, x_2, P(x_3)] + \lambda^2 [x_1, x_2, x_3])$$

$P$ is called a Rota-Baxter operator of weight $\lambda$, and $(A, [ , , ], P)$ is called a Rota-Baxter 3-Lie algebra. When $\lambda = 0$, we have

$$[P(x_1), P(x_2), P(x_3)] = P([P(x_1), P(x_2), x_3] + [P(x_1), x_2, P(x_3)] + [x_1, P(x_2), P(x_3)]). \quad (2)$$

2 Main results

In this section we study the Rota-Baxter operators on the simple 3-Lie algebras over the complex field $F$. From paper [9], there exists only one simple 3-Lie algebra, that is, 4-dimensional 3-Lie algebra $A$ in the following multiplication

$$\begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_3, \\ [e_1, e_3, e_4] = e_2, \\ [e_2, e_3, e_4] = e_1, \end{cases} \quad (3)$$

where $e_1, e_2, e_3, e_4$ is a basis of the 3-Lie algebra $A$.

Let $P : A \rightarrow A$ be a linear mapping. Set $P(e_i) = \sum_{j=1}^{4} a_{ij} e_j, a_{ij} \in F, 1 \leq i, j \leq 4$. Then the matrix form of $P$ in the basis $e_1, e_2, e_3, e_4$ is

$$M(P) = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$
The rank of the matrix \( M(P) \) is called the rank of \( P \) and is denoted by \( R(P) \).

**Theorem** There does not exist Rota-Baxter operators \( P \) with \( R(P) = 3 \) of weight zero on the simple 3-Lie algebra.

**Proof** By Eq. (2) and (3), for \( 1 \leq l < m < n \leq 4 \), we have

\[
[P(e_l), P(e_m), P(e_n)] = \left[ \sum_{j=1}^{4} a_{lj} e_j, \sum_{j=1}^{4} a_{mj} e_j, \sum_{j=1}^{4} a_{nj} e_j \right]
\]

\[
= (a_{l2} a_{m2} a_{n3} - a_{l1} a_{m3} a_{n2}) e_4 + (a_{l1} a_{m2} a_{n4} - a_{l1} a_{m4} a_{n2}) e_3 + (a_{l1} a_{m3} a_{n4} - a_{l1} a_{m4} a_{n3}) e_2 \\
+ (a_{l2} a_{m3} a_{n4} - a_{l2} a_{m4} a_{n3}) e_1 + (a_{l2} a_{m4} a_{n1} - a_{l2} a_{m1} a_{n4}) e_3 + (a_{l2} a_{m3} a_{n1} - a_{l2} a_{m1} a_{n3}) e_4 \\
+ (a_{l3} a_{m4} a_{n2} - a_{l3} a_{m2} a_{n4}) e_1 + (a_{l3} a_{m4} a_{n1} - a_{l3} a_{m1} a_{n4}) e_2 + (-a_{l3} a_{m2} a_{n1} + a_{l3} a_{m1} a_{n2}) e_4 \\
+ (a_{l4} a_{m2} a_{n3} - a_{l4} a_{m3} a_{n2}) e_1 + (a_{l4} a_{m3} a_{n1} - a_{l4} a_{m1} a_{n3}) e_2 + (-a_{l4} a_{m2} a_{n1} + a_{l4} a_{m1} a_{n2}) e_3
\]

\[
P([P(e_l), P(e_m), P(e_n)] + [P(e_l), e_m, P(e_n)] + [e_l, P(e_m), P(e_n)]) = P\left[ \sum_{j=1}^{4} a_{lj} e_j, \sum_{j=1}^{4} a_{mj} e_j, \sum_{j=1}^{4} a_{nj} e_j \right]
\]

\[
+ P\left[ e_l, \sum_{j=1}^{4} a_{mj} e_j, \sum_{j=1}^{4} a_{nj} e_j \right] = P\left( a_{l2} a_{m2} a_{n4} - a_{l1} a_{m4} a_{n3} + a_{l4} a_{m4} a_{n3} \right) e_1 + (a_{l1} a_{m2} a_{n1} - a_{l1} a_{m1} a_{n4} + a_{l4} a_{m1} a_{n4} - a_{l4} a_{m4} a_{n3}) e_2 \\
+ (a_{l1} a_{m3} a_{n4} - a_{l3} a_{m4} a_{n1} + a_{l3} a_{m4} a_{n1} - a_{l1} a_{m4} a_{n3}) e_3 + (a_{l2} a_{m3} a_{n2} - a_{l2} a_{m2} a_{n1} + a_{l1} a_{m3} \\
- a_{l3} a_{m1} + a_{m2} a_{n3} e_4 - a_{m3} a_{n2}) e_4
\]

\[
= (a_{l4} a_{m2} a_{n4} - a_{l2} a_{m4} a_{n3} + a_{l4} a_{m4} a_{n3}) \sum_{j=1}^{4} a_{lj} e_j + (a_{l1} a_{m4} a_{n1} - a_{l4} a_{m1} a_{n4} + a_{m3} a_{n4} \\
- a_{m4} a_{n3}) \sum_{j=1}^{4} a_{2j} e_j + (a_{l1} a_{m4} a_{n1} - a_{l4} a_{m1} a_{n1} + a_{m2} a_{n4} - a_{m4} a_{n2}) \sum_{j=1}^{4} a_{3j} e_j \\
+ (a_{l1} a_{m2} a_{n1} - a_{l2} a_{m1} a_{n1} + a_{m2} a_{n3} e_4 - a_{m3} a_{n2}) \sum_{j=1}^{4} a_{4j} e_j.
\]

Since \( R(P) = 3 \), without loss of generality, we may suppose \( P(e_1), P(e_2), P(e_3) \) are linearly independent. Then vectors

\[
\alpha_1 = (a_{11}, a_{12}, a_{13}, a_{14}), \quad \alpha_2 = (a_{21}, a_{22}, a_{23}, a_{24}), \quad \alpha_3 = (a_{31}, a_{32}, a_{33}, a_{34}),
\]

are linearly independent.

If \( P(e_4) = 0 \).

Then \( [P(e_1), P(e_2), P(e_4)] = P([P(e_1), P(e_2), e_4]) = 0 \), we obtain

\[
P([P(e_1), P(e_2), e_4]) = P\left( \begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right) e_4 + \left( \begin{array}{ccc} a_{11} & a_{13} & a_{14} \\ a_{21} & a_{23} & a_{24} \end{array} \right) e_2 + \left( \begin{array}{ccc} a_{12} & a_{13} & a_{14} \\ a_{22} & a_{23} & a_{24} \end{array} \right) e_1 = 0.
\]
From $[P(e_1), P(e_3), P(e_4)] = [P(e_2), P(e_3), P(e_4)] = 0$, we get

$$P([P(e_1), P(e_3), e_4]) = P\left( \begin{array}{cc} a_{11} & a_{12} \\ a_{31} & a_{32} \end{array} \right) e_3 + \left( \begin{array}{cc} a_{11} & a_{13} \\ a_{31} & a_{33} \end{array} \right) e_2 + \left( \begin{array}{cc} a_{12} & a_{13} \\ a_{32} & a_{33} \end{array} \right) e_1 = 0,$$

$$P([P(e_1), P(e_2), e_4]) = P\left( \begin{array}{cc} a_{21} & a_{22} \\ a_{31} & a_{32} \end{array} \right) e_3 + \left( \begin{array}{cc} a_{21} & a_{23} \\ a_{31} & a_{33} \end{array} \right) e_2 + \left( \begin{array}{cc} a_{22} & a_{23} \\ a_{32} & a_{33} \end{array} \right) e_1 = 0.$$  

Since $P(e_1), P(e_2), P(e_3)$ are linearly independent, we get

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = 0, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = 0, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = 0,$$  

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = 0, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = 0, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = 0.$$  

Therefore, vectors $\alpha_1, \alpha_2, \alpha_3$ are linearly dependent. Contradiction.

Therefore, $P(e_4) \neq 0$. So we might assume that

$$P(e_4) = P(e_1) + \lambda P(e_2) + \mu P(e_3), \lambda, \mu \in F.$$  

Denotes $e'_4 = e_4 - e_1 - \lambda e_2 - \mu e_3$ then $P(e'_4) = 0$. Then by Eq.(2),

$$[P(e_1), P(e_2), P(e_4)] = \mu[P(e_1), P(e_2), P(e_3)]$$

$$= P([P(e_1), P(e_2), e_4] + \lambda[P(e_1), e_2, P(e_2)] + \mu[P(e_1), e_2, P(e_3)]$$

$$+ [e_1, P(e_2), P(e_1)] + \mu[e_1, P(e_2), P(e_3)]).$$

We obtain $P([P(e_1), P(e_2), e'_4]) = 0$. It follows

$$[P(e_1), P(e_2), e'_4] = \kappa_1 e'_4, \kappa_1 \in F.$$  

Similarly, by the direct computation from $[P(e_1), P(e_3), P(e_4)]$ and $[P(e_2), P(e_3), P(e_4)]$, we get

$$[P(e_1), P(e_3), e'_4] = \kappa_2 e'_4, \quad [P(e_2), P(e_3), e'_4] = \kappa_3 e'_4, \kappa_2, \kappa_3 \in F.$$  

Summarizing above discussion, we get that the dimension of the subalgebra generated by the vectors \{ $P(e_1), P(e_2), P(e_3), e'_4$ \} is less than 3. Therefore, vectors $e'_4, P(e_1), P(e_2), P(e_3)$ are linearly dependent then

$$e'_4 = \lambda_1 P(e_1) + \lambda_2 P(e_2) + \lambda_3 P(e_3).$$

So $e'_4$ is contained in the image of $P$. From $P(e'_4) = 0$, $e'_4 \in \text{Ker} P$. It contradicts to $e'_4 \neq 0$.

Therefore, $R(P) \neq 3$. The result follows.
**Remark** There exist Rota-Baxter operators $P$ of weight zero with $R(P) = 1, 2, 3$ on the simple 3-Lie algebra, respectively. For example. Define $P_1, P_2, P_3 : A \rightarrow A$ by

- $P_1(e_1) = e_1 + e_2 + e_3 + e_4$, $P_1(e_2) = P_1(e_3) = P_1(e_4) = 0$.
- $P_2(e_1) = e_1 + e_2$, $P_2(e_2) = e_3 + e_4$, $P_2(e_3) = P_2(e_4) = 0$.
- $P_3(e_1) = e_2$, $P_3(e_2) = -e_1$, $P_3(e_3) = e_4$, $P_3(e_4) = e_3$.

By the direct computation, $P_1, P_2, P_3$ are Rota-Baxter operators, with $R(P_1) = 1$, $R(P_2) = 2$, $R(P_3) = 4$, respectively.

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**References**


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